Problem 1.

A colony $y(t)$ of yeast is growing in a bakery according to the differential equation

$$\frac{dy}{dt} = y^2(y^2 - 9), \; y(0) = y_0 > 0.$$

(i) Find the critical solutions, indicate their type, draw the phase line and sketch the graphs of some solutions.

(ii) For what initial values $y_0 > 0$ will the yeast colony eventually die out?

`Solution:`

(i) *This is an autonomous differential equation. The critical points are found by setting*

$$f(y) = y^2(y^2 - 9) = 0 \implies y = 0 \text{ or } y = \pm 3.$$

We determine the sign of

$$f(y) = y^2(y^2 - 9)$$

as follows:

$$y < -3 \implies y^2(y^2 - 9) > 0$$
$$-3 < y < 0 \text{ or } 0 < y < 3 \implies y^2(y^2 - 9) < 0$$
$$y > 3 \implies y^2(y^2 - 9) > 0.$$

Thus -3 *is an asymptotically stable critical point,* 0 *is semistable, while* 3 *is unstable critical point.*

(ii) *We need*

$$\lim_{t \to \infty} y(t) = 0.$$

If $0 < y_0 < 3$*, solutions will converge to the critical point* 0 *(while for* $y_0 \geq 3$ *solutions will diverge to infinity).*

Problem 2.

Solve the initial value problem: $y' = 1 + 2xy,\ \ y(0) = 1$. (Your answer will require a definite integral.)

`Solution:` *We use integrating factors. We have*

$$y' - 2xy = 1,$$

and the integrating factor is

$$u = \exp^{\int -2x\,dx} = e^{-x^2}.$$

We multiply both sides by the integrating factor u to find

$$(e^{-x^2}y)' = e^{-x^2}.$$

Integrating we find

$$e^{-x^2}y = \int_0^x e^{-t^2}\,dt + C \implies y = e^{x^2}\int_0^x e^{-t^2}\,dt + Ce^{x^2}.$$

Using the initial condition

$$y(0) = 1 \implies C = 1.$$

Hence

$$y = e^{x^2}\int_0^x e^{-t^2}\,dt + e^{x^2}.$$

Problem 3.

Using undetermined coefficients, find the general solution of the differential equation

$$y'' - 2y' + 2y = 5\sin t.$$

`Solution:` *We solve the homogeneous equation*

$$y'' - 2y' + 2y = 0$$

by solving the characteristic equation

$$r^2 - 2r + 2 = 0 \implies r = 1 \pm i.$$

We find

$$y_1 = e^t \cos t,\ y_2 = e^t \sin t.$$

Next, we look for a particular solution in the form

$$y = A\cos t + B\sin t.$$

We calculate

$$y' = B\cos t - A\sin t$$
$$y'' = -A\cos t - B\sin t.$$

Hence

$$y'' - 2y' + 2y = (A - 2B)\cos t + (B + 2A)\sin t = 5\sin t.$$

Then

$$A = 2B,\ B + 2A = 5 \implies A = 2, B = 1.$$

Thus

$$y_p = 2\cos t + \sin t.$$

We find the general solution

$$y = 2\cos t + \sin t + C_1 e^t \cos t + C_2 e^t \sin t.$$

Problem 4.

Consider the differential equation

$$t^2y'' - 3ty' + 3y = 0, \text{ for } t > 0.$$

(i) Find the values of r such that $y = t^r$ is a solution to the differential equation.
(ii) Check that $y_1 = t$ and $y_2 = t^3$ form a fundamental pair of solutions.
(iii) Find the general solution of the differential equation

$$t^2y'' - 3ty' + 3y = t^3 \ln t.$$

Solution:

(i) *We have*

$$y = t^r \implies y' = rt^{r-1},\ y'' = r(r-1)t^{r-2}.$$

Substituting we find

$$t^2y'' - 3ty' + 3y = t^2 \cdot r(r-1)t^{r-2} - 3t \cdot rt^{r-1} + 3t^r = (r^2 - 4r + 3)t^r = 0$$

$$\implies r^2 - 4r + 3 = 0 \implies r = 1,\ r = 3.$$

(ii) *We calculate*

$$W(y_1, y_2) = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 2t^3 \neq 0$$

hence y_1, y_2 form a fundamental pair of solutions for $t > 0$.

(iii) *The homogeneous solution is*

$$y_h = C_1 t + C_2 t^3.$$

We use variation of parameters to find the particular solution. We first write the equation in standard form

$$y'' - \frac{3}{t}y' + \frac{3}{t^2}y = t \ln t.$$

Using integration by parts we find

$$u_1 = -\int \frac{t \ln t \cdot t^3}{2t^3}\, dt = -\frac{1}{2}\int t \ln t\, dt$$

$$= -\frac{1}{2}\left(\frac{1}{2}t^2 \ln t - \int \frac{1}{2}t^2\, d\ln t\right) = -\frac{1}{4}t^2 \ln t + \frac{1}{4}\int t^2 \cdot \frac{1}{t}\, dt = -\frac{1}{4}t^2 \ln t + \frac{1}{4} \cdot \frac{t^2}{2}.$$

Next,

$$u_2 = \int \frac{t \ln t \cdot t}{2t^3}\, dt = \int \frac{\ln t}{2t}\, dt = \frac{1}{4}(\ln t)^2.$$

We conclude

$$y_p = u_1 y_1 + u_2 y_2 = \left(-\frac{1}{4}t^2 \ln t + \frac{1}{4} \cdot \frac{t^2}{2}\right) \cdot t + \frac{1}{4}(\ln t)^2 \cdot t^3 = -\frac{1}{4}t^3 \ln t + \frac{t^3}{8} + \frac{(\ln t)^2 \cdot t^3}{4}.$$

Therefore

$$y = y_h + y_p = C_1 t + C_2 t^3 - \frac{1}{4}t^3 \ln t + \frac{t^3}{8} + \frac{(\ln t)^2 \cdot t^3}{4}.$$

Rearranging constants, this can be rewritten as

$$y = c_1 t + c_2 t^3 - \frac{1}{4}t^3 \ln t + \frac{t^3 (\ln t)^2}{4},$$

for $c_1 = C_1$ and $c_2 = C_2 - \frac{1}{8}$.

Problem 5.

Using the Laplace transform, solve the initial value problem

$$y'' - 2y' + y = t^{10}e^t, \;\; y(0) = 1, y'(0) = 1.$$

`Solution:` *Write $Y(s)$ for the Laplace transform of the solution y. We Laplace transform*

$$y'' - 2y' + y = t^{10}e^t$$

into

$$s^2Y(s) - s - 1 - 2(sY(s) - 1) + Y(s) = \frac{10!}{(s-1)^{11}}.$$

Rearranging terms, we obtain

$$(s-1)^2Y(s) - (s-1) = \frac{10!}{(s-1)^{11}} \implies Y(s) - \frac{1}{s-1} = \frac{10!}{(s-1)^{13}},$$

after dividing by $(s-1)^2$. We now use the inverse Laplace transform to find

$$y(t) - e^t = \frac{1}{12 \cdot 11} \cdot t^{12}e^t \implies y(t) = e^t + \frac{t^{12}e^t}{132}.$$

Problem 6.

The general solution of a certain first order system of differential equations $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{at} \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

where a is a *non-zero* real number.

(i) For what values of a is the origin a (proper) node? Will it be a source or a sink?
(ii) For what values of a is the origin a saddle equilibrium point? Carefully, draw the trajectories in this case.
(iii) For $a = 2$, find the matrix exponential e^{At}.

`Solution:`

(i) *A proper node corresponds to real distinct eigenvalues of the same sign. The eigenvalues are* 1 *and* a. *Thus we need* $a > 0$ *and* $a \neq 1$. *The origin will be a source.*
(ii) *A saddle corresponds to eigenvalues of opposite signs. Thus we need* $a < 0$.
(iii) *We have*

$$\Psi(t) = \begin{bmatrix} e^t & -2e^{2t} \\ 2e^t & e^{2t} \end{bmatrix}.$$

Thus

$$\Psi(0) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \Longrightarrow \Psi(0)^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

We calculate

$$e^{At} = \Psi(t) \cdot \Psi(0)^{-1} = \frac{1}{5} \begin{bmatrix} e^t & -2e^{2t} \\ 2e^t & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} e^t + 4e^{2t} & 2e^t - 2e^{2t} \\ 2e^t - 2e^{2t} & 4e^t + e^{2t} \end{bmatrix}.$$

Problem 7.

Find the general real-valued solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \mathbf{x}.$$

`Solution:` *We first find the eigenvalues and eigenvectors. We have*

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix}$$

whose determinant equals

$$(1-\lambda)(3-\lambda) + 2 = 0 \implies \lambda^2 - 4\lambda + 5 = 0 \implies \lambda = 2 \pm i.$$

We use only one of the eigenvalues, say $\lambda = 2 + i$. We find the eigenvector

$$(A - (2+i)I)\vec{v} = 0 \implies \begin{bmatrix} -1+i & 1 \\ -2 & 1-i \end{bmatrix} \vec{v} = 0 \implies \vec{v} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}.$$

We calculate the complex valued solution

$$\vec{x}_c = e^{(2+i)t} \cdot \begin{bmatrix} 1-i \\ 2 \end{bmatrix} = e^{2t}(\cos t + i \sin t) \cdot \begin{bmatrix} 1-i \\ 2 \end{bmatrix} = e^{2t} \begin{bmatrix} \cos t + \sin t + i(\sin t - \cos t) \\ 2\cos t + 2i \sin t \end{bmatrix}.$$

Taking the real and imaginary parts, we find

$$\vec{x}_1 = e^{2t} \begin{bmatrix} \cos t + \sin t \\ 2\cos t \end{bmatrix}, \ \vec{x}_2 = e^{2t} \begin{bmatrix} \sin t - \cos t \\ 2 \sin t \end{bmatrix}.$$

Then

$$x = c_1 \vec{x}_1 + c_2 \vec{x}_2 = c_1 e^{2t} \begin{bmatrix} \cos t + \sin t \\ 2\cos t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin t - \cos t \\ 2 \sin t \end{bmatrix}.$$

This is not the only possible form of the answer.

Problem 8.

Consider the differential equation

$$y'' + 2xy' + 2y = 0$$

whose solutions are power series in x centered at $x_0 = 0$.

(i) Find the recurrence relation between the coefficients of the power series y.
(ii) Write down the first three *non-zero* terms in each of the two linearly independent solutions.
(iii) What is the radius of convergence of the solutions which contains only even powers of x?

`Solution:`

(i) *We write*

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

We calculate

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \implies 2xy' = \sum_{n=0}^{\infty} 2n a_n x^n,$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

by shifting $n \to n+2$ in the last sum. We compute

$$y'' + 2xy' + 2y = \sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2na_n + 2a_n\right] x^n = 0.$$

Therefore

$$(n+1)(n+2)a_{n+2} + 2na_n + 2a_n = 0 \implies (n+1)(n+2)a_{n+2} + 2(n+1)a_n = 0$$
$$\implies (n+2)a_{n+2} + 2a_n = 0.$$

(ii) *For $n = 0$, we obtain*

$$2a_2 + 2a_0 = 0 \implies a_2 = -a_0.$$

For $n = 2$, we have

$$4a_4 + 2a_2 = 0 \implies a_4 = -\frac{a_2}{2} = \frac{a_0}{2}.$$

Similarly, for $n = 1$, we have

$$3a_3 + 2a_1 = 0 \implies a_3 = -\frac{2}{3}a_1.$$

For $n = 3$, we have

$$5a_5 + 2a_3 = 0 \implies a_5 = -\frac{2}{5}a_3 = \frac{4}{15}a_1.$$

Clearly, a_0 determines all the even power coefficients, and a_1 determines the odd power coefficients. Then the general solution can be written in the form

$$y = a_0\left(1 - x^2 + \frac{x^4}{2} + \dots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \dots\right).$$

The two linearly independent solutions are

$$y_1 = 1 - x^2 + \frac{x^4}{2} + \dots$$

and

$$y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \ldots.$$

(iii) *For the even power solution*

$$y_1 = \sum_k a_{2k} x^{2k}$$

we use the ratio test. We calculate

$$\rho = \lim_{k\to\infty} \left| \frac{a_{2k+2}x^{2k+2}}{a_{2k}x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{-2}{2k+2} x^2 \right| = 0 < 1.$$

To simplify the fraction we used the recurrence found in (i):

$$(2k+2)a_{2k+2} + 2a_{2k} = 0 \implies \frac{a_{2k+2}}{a_{2k}} = -\frac{2}{2k+2}.$$

Thus by the ratio test, we always have convergence or equivalently, the radius of convergence is infinite.

Problem 9.

(i) Find the inverse Laplace transform of the function

$$\frac{1}{(s+1)(s^2+4s+5)}.$$

(ii) Using the Laplace transform, solve the initial value problem

$$y'' + 4y' + 5y = e^{-t} + e^{-t+\pi}u_\pi(t), \quad y(0) = 0, y'(0) = 0.$$

`Solution:`

(i) *We use partial fractions to write*

$$\frac{1}{(s+1)(s^2+4s+5)} = \frac{A}{s+1} + \frac{B(s+2)+C}{s^2+4s+5}.$$

We solve

$$A(s^2+4s+5) + B(s+1)(s+2) + C(s+1) = 1.$$

From here

$$A = \frac{1}{2},\ B = -\frac{1}{2},\ C = -\frac{1}{2}.$$

Thus the fraction becomes

$$\frac{1}{(s+1)(s^2+4s+5)} = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\cdot\frac{(s+2)}{(s+2)^2+1} - \frac{1}{2}\cdot\frac{1}{(s+2)^2+1}.$$

The inverse Laplace transform is

$$\frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t.$$

(ii) *We Laplace transform the differential equation*

$$y'' + 4y' + 5y = e^{-t} + e^{-t+\pi}u_\pi(t)$$

into

$$s^2Y(s) + 4Y(s) + 5Y(s) = \frac{1}{s+1} + \frac{e^{-s\pi}}{s+1} \implies Y(s) = \frac{1}{(s+1)(s^2+4s+5)} + \frac{e^{-s+\pi}}{(s+1)(s^2+4s+5)}.$$

Using the previous part, we calculate

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t + u_\pi(t)\left(\frac{1}{2}e^{-t+\pi} - \frac{1}{2}e^{-2t+2\pi}\cos(t-\pi) - \frac{1}{2}e^{-2t+2\pi}\sin(t-\pi)\right)$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t + u_\pi(t)\left(\frac{1}{2}e^{-t+\pi} + \frac{1}{2}e^{-2t+2\pi}\cos t + \frac{1}{2}e^{-2t+2\pi}\sin t\right).$$

Problem 10.

Two tanks A and B initially contain 2 gallons of fresh water. Water containing 2 lb salt/gallon flows into tank A at a rate of 3 gallons/minute. At the same time, water is drained from tank B at a rate of 3 gallon/minute.

The two tanks are connected by two pipes which allow water to flow in only one direction. Specifically, the first pipe allows water to flow from tank A into tank B at a rate of 4 gallons/minute. The second pipe allows water to flow from tank B into tank A at a rate of 1 gallon/minute.

(i) Let $Q_1(t)$ and $Q_2(t)$ be the quantities of salt (measured in pounds) in tanks A and B at time t. Show that

$$\mathbf{Q}' = \begin{bmatrix} -2 & \frac{1}{2} \\ 2 & -2 \end{bmatrix} \mathbf{Q} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

(ii) Solve the system of differential equations (i) and determine the quantities $Q_1(t)$ and $Q_2(t)$ of salt present in each tank at time t. (Do not forget to take into account the initial conditions.) How much salt will each tank contain as time $t \to \infty$?

`Solution:`

(i) *Begin by drawing a picture. We use that*

$$dQ/dt = c_{in} \cdot rate_{in} - c_{out} \cdot rate_{out}.$$

Consider tank A:

- *there is inflow of salt contributing* $2lb/gal \cdot 3gal/min = 6$ *lb salt/minute,*
- *there is inflow of salt from tank B which contributes* $1 \cdot \frac{Q_2}{2}$ *lb salt/min,*
- *there is outflow of salt to tank B which contributes negatively* $4 \cdot Q_1/2 = 2Q_1$ *lb salt/min.*

Putting everything together

$$\frac{dQ_1}{dt} = 6 + \frac{Q_2}{2} - 2Q_1.$$

Now consider tank B:

- *there is inflow of water from tank A which contributes* $4 \cdot Q_1/2 = 2Q_1$ *lb/min salt,*
- *there is outflow of salt from tank B into tank A which contributes* $1 \cdot Q_2/2$ *lb salt/min,*
- *finally, salt is drained out of tank B, contributing negatively* $3 \cdot Q_2/2$ *lb salt/min.*

Putting things together

$$\frac{dQ_2}{dt} = \frac{1}{2}Q_2 + \frac{3}{2}Q_2 - 2Q_1 = 2Q_2 - 2Q_1.$$

The two equations above can be written in vector form

$$\vec{Q}' = \begin{bmatrix} -2 & \frac{1}{2} \\ 2 & -2 \end{bmatrix} \vec{Q} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

The initial condition is

$$\vec{Q}(0) = 0.$$

(ii) *We find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} -2 & \frac{1}{2} \\ 2 & -2 \end{bmatrix}.$$

Then

$$A - \lambda I = \begin{bmatrix} -2 - \lambda & \frac{1}{2} \\ 2 & -2 - \lambda \end{bmatrix}.$$

The determinant is $(-2-\lambda)^2 - 1 = 0$. *We find*

$$\lambda_1 = -1,\ \lambda_2 = -3.$$

We find the eigenvectors

$$(A+I)\vec{v}_1 = 0 \implies \begin{bmatrix} -1 & \frac{1}{2} \\ 2 & -1 \end{bmatrix} \vec{v}_1 = 0 \implies \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(A+3I)\vec{v}_2 = 0 \implies \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix} \vec{v}_2 = 0 \implies \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We find

$$\vec{Q}_h = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We look for a particular solution Q_p. *In fact, undetermined coefficients suggests that we look for* $\vec{Q}_p$ *as a constant solution which means*

$$\vec{Q}_p' = 0 \implies A\vec{Q}_p + \begin{bmatrix} 6 \\ 0 \end{bmatrix} = 0 \implies \vec{Q}_p = -A^{-1} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -2 & -\frac{1}{2} \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

The general solution is

$$\vec{Q} = \vec{Q}_p + \vec{Q}_h = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We now use the initial condition $\vec{Q}(0) = 0$ *to find the constants* c_1 *and* c_2. *We obtain*

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} 4 \\ 4 \end{bmatrix} \implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

Therefore

$$\vec{Q} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - 3e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus

$$Q_1(t) = 4 - 3e^{-t} - e^{-3t}$$
$$Q_2(t) = 4 - 6e^{-t} + 2e^{-3t}.$$

Clearly

$$Q_1(t) \to 4, Q_2(t) \to 4$$

as $t \to \infty$.