Problem 1. [*12 points.*]

Consider the linear first order equation

$$t^2y' + 3ty = 2e^{t^2}.$$

(i) [*4 points.*] Compute an integrating factor for the differential equation.

We bring the equation into standard linear form

$$y' + \frac{3}{t}y = \frac{2e^{t^2}}{t^2}.$$

We find the integrating factor

$$u = e^{\int \frac{3}{t}\,dt} = e^{3\ln t} = t^3.$$

(ii) [*4 points.*] Find the general solution.

We have

$$(uy)' = u \cdot \frac{2e^t}{t^2} \implies (t^3y)' = 2te^{t^2} \implies t^3y = e^{t^2} + C$$

$$y = \frac{e^{t^2} + C}{t^3}.$$

(iii) [*4 points.*] Find the solution which satisfies the initial condition $y(1) = 0$. What is the maximal interval where the solution is defined?

We solve

$$y(1) = e + C = 0 \implies C = -e.$$

Hence

$$y = \frac{e^{t^2} - e}{t^3}.$$

The solution is defined over the interval $(0, \infty)$.

Problem 2. [*10 points.*]

A tank originally contains 10 gallons of fresh water. Water containing 3 lb of salt per gallon is poured into the tank at a rate of 2 gal/min. The mixture is allowed to leave the tank at the same rate.

(i) [*5 points.*] Write down the differential equation for the amount $Q(t)$ of salt in the tank at time t.

We use that $dQ/dt =$ rate in $-$ rate out, and that rate of salt $=$ rate of water $\cdot$ concentration of salt. Hence

$$\frac{dQ}{dt} = 2 \cdot 3 - 2 \cdot \frac{Q}{10} = 6 - \frac{Q}{5}.$$

(ii) [*5 points.*] Find the amount of salt in the tank after 10 minutes.

We solve by separation of variables

$$\frac{dQ}{dt} = \frac{30 - Q}{5} \implies \frac{dQ}{30 - Q} = \frac{dt}{5} \implies -\ln(30 - Q) = \frac{t}{5} + K \implies$$

$$Q(t) = 30 - Ce^{-t/5}.$$

Since $Q(0) = 0$, we obtain $C = 30$ hence

$$Q(t) = 30(1 - e^{-t/5}) \implies Q(10) = 30(1 - e^{-2}).$$

Problem 3. [*10 points.*]

Consider the differential equation
$$(3x^2 + y^2) + (2xy + 1)y' = 0.$$

(i) [*4 points.*] Explain why the differential equation is exact.

We set
$$M = 3x^2 + y^2, N = 2xy + 1.$$

We calculate
$$M_y = 2y,\ N_x = 2y \implies M_y = N_x \implies \text{exact equation.}$$

(ii) [*6 points.*] Solve the differential equation. It suffices to give the solution implicitly.

We look for a potential function
$$f_x = 3x^2 + y^2,\ f_y = 2xy + 1.$$

We integrate the first equation
$$f = x^3 + xy^2 + h(y)$$

and substitute into the second
$$f_y = 2xy + h'(y) = 2xy + 1 \implies h'(y) = 1 \implies h(y) = y.$$

Thus
$$f = x^3 + xy^2 + y$$

and the implicit solution is
$$x^3 + xy^2 + y = C.$$

Problem 4. [*10 points.*]

Consider the autonomous equation

$$\frac{dy}{dt} = 4y - y^2.$$

(i) [*7 points.*] Determine the critical points and indicate their type i.e. asymptotically stable, unstable, semistable. Sketch the phase line.

The critical points are found by solving

$$4y - y^2 = 0 \implies y = 0,\ y = 4.$$

We have

$$4y - y^2 < 0, \text{ for } y < 0,\ 4y - y^2 > 0 \text{ for } 0 < y < 4, \text{ and } 4y - y^2 < 0 \text{ for } y > 4.$$

Therefore $y = 0$ is unstable, $y = 4$ is asymptotically stable.

(ii) [*3 points.*] What is the long-term behavior of the solution satisfying the initial value $y(0) = 2$?

Since $0 < y(0) < 2$, we have $y(t) \to 4$ as $t \to \infty$.

Problem 5. [*10 points.*]

Find the general solution of the differential equation $y'' + 4y' + 13y = 0$.

We form the characteristic equation

$$r^2 + 4r + 13 = 0$$

which gives $r_{1,2} = -2 \pm 3i$. We obtain the fundamental pair

$$y_1 = e^{-2t} \cos 3t, \;\; y_2 = e^{-2t} \sin 3t$$

hence

$$y = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t).$$

Problem 6. [*8 points.*]

Consider the differential equation

$$y'' + 2ty' + q(t)\,y = 0,$$

for some unknown function $q(t)$.

Two solutions y_1 and y_2 of the differential equation satisfy the initial conditions

$$y_1(0) = 1,\;\; y_2(0) = 2$$
$$y_1'(0) = -1,\;\; y_2'(0) = 3.$$

(i) [*4 points*] Determine the Wronskian $W(y_1, y_2)$ as a function of t. Do y_1 and y_2 form a fundamental pair of solutions?

By Abel's theorem

$$W(y_1, y_2) = C\exp\left(-\int 2tdt\right) = C\exp(-t^2).$$

The initial conditions give

$$W(y_1, y_2)(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = 5$$

hence $C = 5$ and

$$W(y_1, y_2) = 5e^{-e^2} \neq 0.$$

Thus y_1, y_2 form a fundamental pair.

(ii) [*4 points*] A third solution satisfies the initial value problem

$$y(0) = 1, y'(0) = 7.$$

Express this solution in terms of y_1 and y_2.

We need $y_3 = c_1y_1 + c_2y_2$ and imposing the initial conditions we have

$$y_3(0) = c_1y_1(0) + c_2y_2(0) \implies c_1 + 2c_2 = 1$$
$$y_3'(0) = c_1y_1'(0) + c_2y_2'(0) \implies -c_1 + 3c_2 = 7.$$

We find

$$c_1 = -\frac{11}{5}, c_2 = \frac{8}{5}$$

hence

$$y_3 = -\frac{11}{5}y_1 + \frac{8}{5}y_2.$$