Problem 1. [*10 points.*]

Solve the following differential equation by variation of parameters

$$y'' - 2y' + y = t^{-2}e^t.$$

We solve the homogeneous equation

$$y'' - 2y' + y = 0.$$

Two solutions are

$$y_1 = e^t,\ y_2 = te^t.$$

We calculate

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t}.$$

We look for solutions $y = u_1y_1 + u_2y_2$. We calculate

$$u_1 = -\int \frac{t^{-2}e^t \cdot te^t}{e^{2t}}\,dt = -\int t^{-1}\,dt = -\ln t + C_1$$

and

$$u_2 = \int \frac{t^{-2}e^t \cdot e^t}{e^{2t}} = \int t^{-2}\,dt = -\frac{1}{t} + C_2.$$

We find

$$y = -\ln t \cdot e^t - \frac{1}{t} \cdot te^t + C_1e^t + C_2te^t$$

which can be rewritten as

$$x = -\ln t \cdot e^t + c_1e^t + c_2te^t$$

for $c_1 = C_1 - 1$ and $c_2 = C_2$.

Problem 2. [*15 points.*]

Using undetermined coefficients, find a particular solution of the differential equation

$$y'' - 8y' + 7y = -e^{2t}(5t + 9).$$

We look for a solution of the form

$$y = e^{2t}(At + B).$$

We find

$$y' = e^{2t}A + 2e^{2t}(At + B) = e^{2t}(2At + 2B + A).$$

Also

$$y'' = e^{2t}(2A) + 2e^{2t}(2At + 2B + A) = e^{2t}(4At + 4B + 4A).$$

We calculate

$$y'' - 8y' + 7y = -e^{2t}(5At + 5B + 4A) = -e^{2t}(5t + 9).$$

Therefore

$$A = 1 \text{ and } 5B + 4A = 9$$

hence $B = 1$. A particular solution is

$$y = e^{2t}(t + 1).$$

Problem 3. [*15 points.*]

(i) [*4 points*] Consider the system

$$\mathbf{x}' = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \mathbf{x}.$$

An eigenvalue of the matrix of coefficients is

$$\lambda = 3 + 4i.$$

(*You do not need to check this fact.*) Without solving the system, carefully sketch the trajectory of the solution which has the initial value $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Be sure to include arrows on your diagram that indicate the direction of the trajectory.

The trajectories are unstable spirals. For the initial value $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we calculate the velocity vector

$$\mathbf{x}'(0) = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

The spiral comes out of the origin clockwise.

(ii) [*5 points*] The defective matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ has a *repeated* eigenvalue $\lambda = 2$ and a corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. (*You do not need to check this fact.*) Write down two independent solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ for the system $\mathbf{x}' = A\mathbf{x}$.

We have

$$\mathbf{x_1} = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For the second solution, we need a generalized eigenvector $\mathbf{u}$ which solves

$$(A - 2I)\mathbf{u} = \mathbf{v} \implies \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can take

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The second solution is

$$\mathbf{x_2} = e^{2t}(t\mathbf{v} + \mathbf{u}) = e^{2t} \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

(iii) [*6 points*] The general solution of a certain first order system of differential equations is

$$\mathbf{x} = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Carefully sketch the corresponding trajectories.

Since the eigenvalues are real distinct and of the same sign, the origin is an unstable node. To sketch the solution, we find the dominant term. As $t \to -\infty$, we have $\mathbf{x} \to 0$ and the term that dominates is $e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. As $t \to \infty$, $x \to \infty$ and the term that dominates is $e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Thus our trajectories are tangent to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ near the origin and follow the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ at infinity.

Problem 4. [*20 points.*]

Consider the system $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}.$$

(i) [*6 points*] Write down the general solution.

We calculate the eigenvalues and eigenvectors. We find

$$A - \lambda I = \begin{bmatrix} -3-\lambda & -2 \\ 4 & 3-\lambda \end{bmatrix}.$$

The determinant equals

$$(-3-\lambda)(3-\lambda) + 8 = 0 \implies \lambda^2 - 1 = 0 \implies \lambda = \pm 1.$$

For the eigenvalue $\lambda = 1$, we have

$$A - I = \begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

For the eigenvalue $\lambda = -1$, we have

$$A + I = \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix} \implies \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The general solution is

$$\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(ii) [*4 points*] Compute the matrix exponential e^{At}.

We find a fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^t & e^{-t} \\ -2e^t & -e^{-t} \end{bmatrix}.$$

We have

$$\Psi(0) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \implies \Psi(0)^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}.$$

Then

$$e^{At} = \Psi(t)\Psi(0)^{-1} = \begin{bmatrix} e^t & e^{-t} \\ -2e^t & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{-t} & -e^t + e^{-t} \\ 2e^t - 2e^{-t} & 2e^t - e^{-t} \end{bmatrix}$$

(iii) [*2 points*] Solve the initial value problem $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We have

$$\mathbf{x} = e^{At}\mathbf{x}(\mathbf{0}) = \begin{bmatrix} -e^t + 2e^{-t} & -e^t + e^{-t} \\ 2e^t - 2e^{-t} & 2e^t - e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{-t} \\ 2e^t - 2e^{-t} \end{bmatrix}.$$

(iv) [*8 points*] Find a particular solution using variation of parameters

$$\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}.$$

We use variation of parameters to find

$$x_p = \Psi(t) \int \Psi(t)^{-1} \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}.$$

We have

$$\Psi(t) = \begin{bmatrix} e^{t} & e^{-t} \\ -2e^{t} & -e^{-t} \end{bmatrix} \implies \Psi(t)^{-1} = \begin{bmatrix} -e^{-t} & -e^{-t} \\ 2e^{t} & e^{t} \end{bmatrix}.$$

Thus

$$\begin{aligned} x_p &= \begin{bmatrix} e^{t} & e^{-t} \\ -2e^{t} & -e^{-t} \end{bmatrix} \int \begin{bmatrix} -e^{-t} & -e^{-t} \\ 2e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} dt = \begin{bmatrix} e^{t} & e^{-t} \\ -2e^{t} & -e^{-t} \end{bmatrix} \int \begin{bmatrix} -e^{t} \\ 2e^{3t} \end{bmatrix} dt \\ &= \begin{bmatrix} e^{t} & e^{-t} \\ -2e^{t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -e^{t} \\ \frac{2}{3}e^{3t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}e^{2t} \\ \frac{4}{3}e^{2t} \end{bmatrix} \end{aligned}$$