Math 20D - Fall 2011 - Final Exam

Problem 1.

A population $y(t)$ of turtles is growing on an island according to the logistic equation with harvesting

$$\frac{dy}{dt} = y(600 - y) - 50,000, \ \ y(0) = y_0 > 0.$$

(i) Find the critical solutions, indicate their type, draw the phase line and sketch the graphs of some solutions.

(ii) Assume that at time $t = 0$ there are 200 turtles on the island. How many turtles will there be on the island in the long run?

`Answer:`

(i) *We find the critical points*

$$\frac{dy}{dt} = y(600 - y) - 50,000 = (-y + 100)(y - 500) = 0 \implies y = 100 \text{ and } y = 500.$$

The parabola $y(600 - y) - 50,000$ *is concave, so the signs are negative for* $y < 100$, *positive for* $100 < y < 500$ *and negative for* $y > 500$. *In particular, the function* y *is decreasing for* $y < 100$, *increasing for* $100 < y < 500$ *and decreasing for* $y > 500$. *Drawing the phase line and sketching some of the solutions, we see that* $y = 100$ *repels solutions hence it is an unstable critical point. On the other hand* $y = 500$ *attracts solutions, hence* $y = 500$ *is a stable critical point.*

(ii) *Since* $y(0) = 200$ *which falls in the interval* $(100, 500)$, *it follows that the solution converges to the stable critical point*

$$\lim_{t \to \infty} y(t) = 500.$$

Problem 2.

Consider the inhomogeneous differential equation

$$(\star)\quad x^2y'' - xy' + y = x\ln x,\ \text{for } x > 0.$$

This problem has three main parts (A), (B), (C), all independent of each other.

(A.) Check that $y_1 = x$ is a solution to the homogeneous differential equation. We now proceed to find a second solution y_2 to the homogeneous equation.

(B.1) Show that for any fundamental pair of solutions (y_1, y_2) to the homogeneous equation we must have $W(y_1, y_2) = Cx$ for some constant $C \neq 0$.

(B.2) Set $y_1 = x$. Consider a second solution y_2 to the the homogeneous equation satisfying the initial values

$$y_2(1) = 0,\ y_2'(1) = 1.$$

Show that $W(y_1, y_2) = x$.

(B.3) Use part (B.2) to show that the solution y_2 must satisfy

$$xy_2' - y_2 = x.$$

(B.4) Use (B3) to find a second solution y_2.

(C) Using the solutions

$$y_1 = x \text{ and } y_2 = x\ln x$$

to the homogeneous equation, find the general solution to the inhomogeneous equation $(\star)$ by variation of parameters.

`Answer:`

(A) *We verify that $y_1 = x$ is a solution by computing $y_1' = 1, y_1'' = 0$. Direct computation then shows that the differential equation is verified*

$$x^2y_1'' - xy_1' + y_1 = 0.$$

(B1) *This follows by Abel's theorem. We first bring the equation in standard form*

$$y'' - \frac{1}{x}y + \frac{1}{x^2}y = 0.$$

Abel's theorem states that

$$W(y_1, y_2) = C\exp\left(\int \frac{1}{x}\,dx\right) = C\exp(\ln x) = Cx$$

as needed.

(B2) *We compute*

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & y_2 \\ 1 & y_2' \end{vmatrix} = xy_2' - y_2.$$

Evaluating at $x = 1$ we find

$$W(y_1, y_2)(1) = 1 \cdot y_2'(1) - y_2(1) = 1$$

using the initial conditions $y_2(1) = 0, y_2'(1) = 1$*. Since we already showed in (B1) that* $W(y_1, y_2) = Cx$ *it follows*

$$W(y_1, y_2)(1) = C \cdot 1 = C$$

from where $C = 1$ *by comparing with the preceding equation. Thus* $W(y_1, y_2) = x$.

(B3) *We showed in part (B2) that*

$$W(y_1, y_2) = xy_2' - y_2 \text{ and } W(y_1, y_2) = x$$

from where the conclusion follows.

(B4) *To find* y_2 *we use integrating factors. We first write the equation* $xy_2' - y_2 = x$ *in standard form*

$$y_2' - \frac{1}{x} y_2 = 1.$$

The integrating factor is

$$\mu = \exp\left(-\int \frac{1}{x}\right) = \exp(-\ln x) = \frac{1}{x}.$$

Multiplying both sides by the integrating factor we find

$$\left(\frac{1}{x} y_2\right)' = \frac{1}{x} \implies \frac{1}{x} y_2 = \ln x + K \implies y_2 = x \ln x + Kx.$$

To find the constant K *we use the initial value* $y_2(1) = 0$ *which yields* $K = 0$ *so that*

$$y_2 = x \ln x.$$

(C) *We bring the equation to be solved into standard form*

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = \frac{\ln x}{x}.$$

We have computed $W(y_1, y_2) = x$ *above. By variation of parameters a particular solution is*

$$y_p = u_1 y_1 + u_2 y_2.$$

We have

$$u_1 = -\int \frac{\ln x}{x} \cdot \frac{y_2}{W}\, dx = -\int \frac{\ln x}{x} \cdot \frac{x \ln x}{x}\, dx = -\int \frac{(\ln x)^2}{x} = -\int (\ln x)^2 \cdot (\ln x)'\, dx = -\frac{1}{3}(\ln x)^3.$$

Similarly,

$$u_2 = \int \frac{\ln x}{x} \cdot \frac{y_1}{W}\, dx = \int \frac{\ln x}{x} \cdot \frac{x}{x}\, dx = \int \frac{\ln x}{x}\, dx = \int (\ln x) \cdot (\ln x)'\, dx = \frac{1}{2}(\ln x)^2.$$

A particular solution is found by substituting into the above expression

$$y_p = -\frac{1}{3}(\ln x)^3 \cdot x + \frac{1}{2}(\ln x)^2 \cdot x \ln x = \frac{1}{6} x (\ln x)^3.$$

The general solution takes the form

$$y = y_p + y_h = y_p + c_1 y_1 + c_2 y_2 = \frac{1}{6} x (\ln x)^3 + c_1 x + c_2 x \ln x.$$

Problem 3.

Consider the system $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} -2 & -8 \\ 1 & -8 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -6$. (You do not need to check this fact.)

(i) Find a fundamental pair of solutions to the system.

(ii) Draw the trajectories of the general solution. What is the type of the phase portrait you obtained?

(iii) Calculate the matrix exponential e^{At}.

(iv) Solve the initial value problem $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(v) Use variation of parameters to find a particular solution the following inhomogeneous system

$$\vec{x}' = Ax + \begin{bmatrix} 12t \\ 0 \end{bmatrix}.$$

`Answer:`

(i) *We find eigenvectors for the two eigenvalues. Letting* $A = \begin{bmatrix} -2 & -8 \\ 1 & -8 \end{bmatrix}$ *we compute for the first eigenvalue*

$$A + 4I = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

For the second eigenvalue, we compute

$$A + 6I = \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} \implies \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We form the two fundamental solutions

$$\vec{x}_1 = e^{-4t} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{x}_2 = e^{-6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(ii) *The general solution is*

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 e^{-4t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

When $t \to -\infty$, the solutions are of large magnitude and follow the dominant term e^{-6t} in the direction of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. When $t ø \infty$, the solutions approach zero, and they follow the dominant term e^{-4t} in the direction $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. The origin is a node sink.

(iii) *We have*

$$e^{At} = \Phi(t) = \Psi(t) \cdot \Psi(0)^{-1}.$$

We find the fundamental matrix

$$\Psi(t) = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix}.$$

Thus

$$\Psi(0) = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \implies \Psi(0)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}.$$

Substituting we find

$$e^{At} = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4e^{-4t} - 2e^{-6t} & -8e^{-4t} + 8e^{-6t} \\ e^{-4t} - e^{-6t} & -2e^{-4t} + 4e^{-6t} \end{bmatrix}.$$

(iv) *We have*

$$\vec{x} = e^{At} \cdot \vec{x}_0 = \frac{1}{2} \begin{bmatrix} 4e^{-4t} - 2e^{-6t} & -8e^{-4t} + 8e^{-6t} \\ e^{-4t} - e^{-6t} & -2e^{-4t} + 4e^{-6t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4e^{-4t} + 6e^{-6t} \\ -e^{-4t} + 3e^{-6t} \end{bmatrix}.$$

(v) *We compute*

$$\vec{x} = \Psi(t) \int \Psi(t)^{-1} \begin{bmatrix} 12t \\ 0 \end{bmatrix} dt.$$

We have

$$\Psi(t)^{-1} = \frac{1}{2e^{-10t}} \begin{bmatrix} e^{-6t} & -2e^{-6t} \\ -e^{-4t} & 4e^{-4t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{4t} & -2e^{4t} \\ -e^{6t} & 4e^{6t} \end{bmatrix}.$$

Thus

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \int \frac{1}{2} \begin{bmatrix} e^{4t} & -2e^{4t} \\ -e^{6t} & 4e^{6t} \end{bmatrix} \cdot \begin{bmatrix} 12t \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \int \begin{bmatrix} 6te^{4t} \\ -6te^{6t} \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2}\left(t - \frac{1}{4}\right) e^{4t} \\ -\left(t - \frac{1}{6}\right) e^{6t} \end{bmatrix} = \begin{bmatrix} 4t - \frac{7}{6} \\ \frac{1}{2}t - \frac{5}{24} \end{bmatrix}. \end{aligned}$$

The integrals were computed via integration by parts. For instance

$$\int 6te^{6t}\, dt = \int t(e^{6t})'\, dt = te^{6t} - \int e^{6t}\, dt = te^{6t} - \frac{1}{6}e^{6t} = \left(t - \frac{1}{6}\right) e^{6t}.$$

The second integral is similar

$$\int 6te^{4t}\, dt = \int \frac{3}{2}t(e^{4t})'\, dt = \frac{3}{2}\left(te^{4t} - \int e^{4t}\, dt\right) = \frac{3}{2}\left(t - \frac{1}{4}\right) e^{4t}.$$

Problem 4.

Find two independent real valued solutions of the system

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \vec{x}.$$

`Answer:` *We write* $A = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$. *We compute Tr* $A = 4, \det A = 8$ *so the characteristic polynomial is*

$$\lambda^2 - 4\lambda + 8 = 0 \implies (\lambda - 2)^2 + 4 = 0 \implies \lambda - 2 = \pm 2i \implies \lambda = 2 \pm 2i.$$

We use only one of the eigenvalues below, say $\lambda = 2 + 2i$. *We find an eigenvector by computing*

$$A - (2+2i)I = A = \begin{bmatrix} 1 - (2+2i) & 1 \\ -5 & 3 - (2+2i) \end{bmatrix} = A = \begin{bmatrix} -1 - 2i & 1 \\ -5 & 1 - 2i \end{bmatrix} \implies \vec{v} = \begin{bmatrix} 1 \\ 1 + 2i \end{bmatrix}.$$

Thus a complex valued solution is given by

$$\vec{x}_1 = e^{(2+2i)t} \begin{bmatrix} 1 \\ 1 + 2i \end{bmatrix} = e^{2t}(\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ 1 + 2i \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} \cos 2t + i \sin 2t \\ (1+2i)(\cos 2t + i \sin 2t) \end{bmatrix} = e^{2t} \begin{bmatrix} \cos 2t + i \sin 2t \\ \cos 2t - 2 \sin 2t + i(2 \cos 2t + \sin 2t) \end{bmatrix}.$$

We find the real valued solutions by taking the real and imaginary part of the complex valued solution. We have

$$u_1 = e^{2t} \begin{bmatrix} \cos 2t \\ \cos 2t - 2 \sin 2t \end{bmatrix}, v_1 = e^{2t} \begin{bmatrix} \sin 2t \\ 2 \cos 2t + \sin 2t \end{bmatrix}.$$

are the real valued solutions. There are other possible answers here as well.

Problem 5.

Consider the differential equation

$$y'' - xy' - y = 0$$

whose solutions are power series in x centered at $x_0 = 0$.

(i) Find the recurrence relation between the coefficients of the power series y.

(ii) Write down the first three *non-zero* terms in each of the two linearly independent solutions.

(iii) Express the solution involving only even powers of x in closed form. The final answer should be a familiar exponential. You may need to recall the series expansion

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots + \frac{y^n}{n!} + \ldots.$$

`Answer:`

(i) *We write*

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

We compute

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \implies xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

where in the above we used that the term corresponding to $n = 0$ is in fact zero $na_n = 0$ for $n = 0$. In addition,

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n,$$

where the shift $n \to n+2$ was done in the last step. Thus

$$\begin{aligned} y'' - xy' - y &= \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left[a_{n+2}(n+1)(n+2) - n a_n - a_n\right] \cdot x^n \\ &= \sum_{n=0}^{\infty} \left[a_{n+2}(n+1)(n+2) - a_n(n+1)\right] \cdot x^n. \end{aligned}$$

Since $y'' - xy' - y = 0$ we conclude

$$a_{n+2}(n+1)(n+2) - a_n(n+1) = 0 \implies a_{n+2}(n+2) - a_n = 0$$

for all n.

(ii) *We write down the first coefficients of the even solution by using $n = 0, n = 2$. We find*

$$2a_2 - a_0 = 0 \implies a_2 = \frac{a_0}{2}$$

$$4a_4 - a_2 = 0 \implies a_4 = \frac{a_2}{4} = \frac{a_0}{2 \cdot 4}.$$

The even solution is

$$y^{even} = a_0 + a_2x^2 + a_4x^4 + \ldots = a_0 + \frac{a_0}{2}x^2 + \frac{a_0}{2\cdot 4}x^4 + \ldots$$
$$= a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \ldots\right).$$

Here we can even set $a_0 = 1$ if we wish to find an answer without any undetermined constants.

For the odd solution we use $n = 1$ and $n = 3$ to find

$$3a_3 - a_1 = 0 \implies a_3 = \frac{a_1}{3}$$
$$5a_5 - a_3 = 0 \implies a_5 = \frac{a_3}{5} = \frac{a_1}{3\cdot 5}.$$

This yields

$$y^{odd} = a_1x + a_3x^3 + a_5x^5 + \ldots = a_1\left(x + \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 + \ldots\right).$$

Again, we could use $a_1 = 1$ if we wish to find an answer without any undetermined constants.

(iii) *We wish to first the pattern for the even solution. If we continue further with $n = 4$ we find*

$$6a_6 - a_4 = 0 \implies a_6 = \frac{a_4}{6} = \frac{a_0}{2\cdot 4\cdot 6}$$

while $n = 6$ yields

$$8a_8 - a_6 = 0 \implies a_8 = \frac{a_6}{8} = \frac{a_0}{2\cdot 4\cdot 6\cdot 8}.$$

The pattern is now clear

$$a_{2k} = \frac{a_0}{2\cdot 4\cdot 6\cdot \ldots \cdot (2k)} = \frac{a_0}{2^k\cdot(1\cdot 2\cdot\cdots\cdot k)} = \frac{a_0}{2^k k!}.$$

Let us set $a_0 = 1$ since we wish to speak about a specific even solution (which is only unique up to scaling). Then

$$a_{2k} = \frac{1}{2^k k!}$$

and

$$y^{even} = \sum_{k=0}^{\infty} a_{2k}x^{2k} = \sum_{k=0}^{\infty} \frac{1}{2^k k!}x^{2k} = \sum_{k=0}^{\infty} \frac{1}{k!}\cdot\left(\frac{x^2}{2}\right)^k = e^{\frac{x^2}{2}}.$$

Problem 6.

Consider the function

$$h(t) = \begin{cases} 0 & t < 1 \\ t^2 & 1 \leq t < 2 \\ t^2 + t - 2 & t \geq 2. \end{cases}$$

(i) Express h in terms of unit step functions.

(ii) Find the Laplace transform of h. You may leave your answer as a sum of fractions.

`Answer:`

(i) *We have* $h(t) = t^2 u_1(t) + (t-2)u_2(t)$.

(ii) *We use that*

$$f(t-c)u_c(t) \mapsto e^{-cs}F(s).$$

In our case, the second term is a direct application (taking $c = 2$ *and* $f(t) = t$ *so that* $F(s) = \frac{1}{s^2}$*), so*

$$(t-2)u_2(t) \mapsto \frac{e^{-2s}}{s^2}.$$

For the first term, we wish to write

$$t^2 u_1(t) = f(t-1)u_1(t)$$

for some suitable function f *in order to apply the formula. This means*

$$f(t-1) = t^2 \implies f(t) = (t+1)^2 = t^2 + 2t + 1 \implies F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}.$$

We have

$$t^2 u_1(t) = f(t-1)u_1(t) \mapsto e^{-s}F(s) = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s}.$$

Therefore

$$H(s) = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2}.$$

Problem 7.

Use Laplace transforms to solve the initial value problem

$$y'' + 2y' + 5y = e^{-2t},\ y(0) = 0, y'(0) = 1.$$

`Answer:` *We have*

$$y'' \mapsto s^2Y - sy(0) - y'(0) = s^2Y - 1,$$
$$y' \mapsto sY - y(0) = sY.$$

The equation to be solved becomes after applying Laplace transform

$$s^2Y - 1 + 2sY + 5Y = \frac{1}{s+2} \implies (s^2 + 2s + 5)Y = 1 + \frac{1}{s+2}$$
$$\implies Y = \frac{1}{s^2 + 2s + 5} + \frac{1}{(s+2)(s^2 + 2s + 5)}.$$

We need to compute the inverse Laplace transforms of the above expression. The first term

$$\frac{1}{s^2 + 2s + 5} = \frac{1}{(s+1)^2 + 4} \text{ has inverse Laplace equal to } \frac{1}{2}\sin 2te^{-t}.$$

The second term is more difficult. We use partial fractions to write

$$\frac{1}{(s+2)(s^2 + 2s + 5)} = \frac{A}{s+2} + \frac{Bs + C}{s^2 + 2s + 5}.$$

Direct computation yields

$$A(s^2 + 2s + 5) + (s+2)(Bs + C) = 1 \iff s^2(A + B) + s(2A + 2B + C) + 5A + 2C = 1$$
$$\iff A + B = 0, 2A + 2B + C = 0, 5A + 2C = 0 \iff A = \frac{1}{5}, B = -\frac{1}{5}, C = 0.$$

Thus

$$\frac{1}{(s+2)(s^2 + 2s + 5)} = \frac{1}{5}\left(\frac{1}{s+2} - \frac{s}{s^2 + 2s + 5}\right) = \frac{1}{5}\left(\frac{1}{s+2} - \frac{s+1}{(s+1)^2 + 4} + \frac{1}{(s+1)^2 + 4}\right).$$

The Laplace inverse equals

$$\frac{1}{5}\left(e^{-2t} - e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t\right).$$

Collecting all terms

$$y(t) = \frac{1}{2}\sin 2te^{-t} + \frac{1}{5}\left(e^{-2t} - e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t\right)$$

or simplifying

$$y(t) = \frac{e^{-2t}}{5} + \frac{3}{5}e^{-t}\sin 2t - \frac{1}{5}e^{-t}\cos 2t.$$

Problem 8.

Consider the forcing function

$$h(t) = u_\pi(t) - u_{4\pi}(t).$$

(i) Solve the following initial value problem using Laplace transform

$$y'' + y = h(t),\ y(0) = y'(0) = 0.$$

(ii) Write your solution $y(t)$ explicitly over each of the three intervals

$$0 \le t < \pi, \quad \pi \le t < 4\pi, \quad 4\pi \le t < \infty.$$

(iii) Draw the graph of the solution you found in (i).

`Answer:`

(i) *Using the Laplace of* $u_c(t) \mapsto \frac{e^{-cs}}{s}$, *we compute*

$$H(s) = \frac{e^{-s\pi}}{s} - \frac{e^{-4\pi s}}{s}.$$

The Laplace transform of the differential equation becomes

$$s^2 Y + Y = H(s) \implies Y = \frac{H(s)}{s^2+1} = \frac{e^{-\pi s} - e^{-4\pi s}}{s(s^2+1)}.$$

We need to find the inverse Laplace transform of this last expression. We first decompose into partial fractions

$$F(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

This is the Laplace transform of the function

$$f(t) = 1 - \cos t.$$

Using that $e^{cs}F(s)$ *has Laplace inverse* $u_c(t)f(t-c)$ *we have*

$$Y = e^{-\pi s}F(s) - e^{-4\pi s}F(s) \implies y = u_\pi(t)f(t-\pi) - u_{4\pi}(t)f(t-4\pi)$$

$$\implies y = u_\pi(t)(1-\cos(t-\pi)) - u_{4\pi}(t)(1-\cos(t-4\pi)).$$

Using periodicity this can be further simplified to

$$y = u_\pi(t)(1+\cos t) - u_{4\pi}(1-\cos t).$$

(ii)
- *For* $t < \pi$ *we have* $u_\pi(t) = u_{4\pi}(t) = 0$ *so* $y = 0$
- *For* $\pi \le t < 4\pi$ *we have* $u_\pi(t) = 1$ *but* $u_{4\pi}(t) = 0$ *so* $y = 1 + \cos t$
- *Finally for* $t > 4\pi$ *we have* $u_\pi(t) = u_{4\pi}(t) = 1$ *so* $y = 1 + \cos t - (1-\cos t) = 2\cos t$.

Thus

$$y(t) = \begin{cases} 0 & \text{if } t < \pi \\ 1 + \cos t & \text{if } \pi \le t < 4\pi \\ 2\cos t & \text{if } t \ge 4\pi \end{cases}.$$