**Problem 1.**

Consider the differential equation

$$(4x^3y^3 + ay^2) + (3x^4y^2 + 2xy + 2)y' = 0.$$

(i) For what values of $a$ is the differential equation exact?

We have

$$M = 4x^3y^3 + ay^2 \implies M_y = 12x^3y^2 + 2ay$$
$$N = 3x^4y^2 + 2xy + 2 \implies N_x = 12x^3y^2 + 2y.$$

Since $M_y = N_x$ we must have $a = 1$.

(ii) Solve the differential equation, when exact. It suffices to give the solution implicitly.

We look for a potential function

$$f_x = 4x^3y^3 + y^2 \implies f = x^4y^3 + xy^2 + h(y) \implies f_y = 3x^4y^2 + 2xy + h'(y).$$

Since

$$f_y = 3x^4y^2 + 2xy + 2$$

we must have $h'(y) = 2$ hence $h(y) = 2y$ hence

$$f = x^4y^3 + xy^2 + 2y.$$

Solutions are obtained by setting

$$x^4y^3 + xy^2 + 2y = C.$$

**Problem 2.**

A colony $y(t)$ is growing in a bakery according to the differential equation

$$\frac{dy}{dt} = y^2 - 5y + 6.$$

Determine the critical points and indicate their type i.e. asymptotically stable, unstable, semistable. Sketch the phase line. What is the long-term behavior of the solution satisfying the initial value $y(0) = 1$?

The critical points are found by solving

$$y^2 - 5y + 6 = 0 \implies (y-2)(y-3) = 0 \implies y = 2 \text{ or } y = 3.$$

We have

$$\begin{gathered} y < 2 \text{ or } y > 3 \implies y^2 - 5y + 6 > 0 \\ 2 < y < 3 \implies y^2 - 5y + 6 < 0. \end{gathered}$$

Thus 2 is an asymptotically stable critical point, while 3 is an unstable critical point. From the phase diagram, we see $\lim_{t\to\infty} y(t) = 2$ since $y(0) < 2$ and 2 is asymptotically stable.

**Problem 3.**

Solve the linear first order equation

$$t^3y' + 4t^2y = 2\sin t.$$

We solve by integrating factor. We have

$$y' + \frac{4}{t}y = \frac{2\sin t}{t^3}.$$

We find

$$u = \exp(\int \frac{4}{t}\,dt) = \exp(4\ln t) = t^4.$$

We multiply both sides by $u$ to obtain

$$(t^4y)' = 2t\sin t.$$

Integrating, we have

$$t^4y = \int 2t\sin t\,dt = \int -2td(\cos t) = -2t\cos t + \int 2\cos t\,dt = -2t\cos t + 2\sin t + C.$$

The last integral was computed by parts. We conclude

$$y = -\frac{2\cos t}{t^3} + \frac{2\sin t}{t^4} + \frac{C}{t^4}.$$

**Problem 4.**

(i) Write down a second order homogeneous constant coefficient differential equation with solution

$$y = e^{3t}(\cos t + 4\sin t).$$

Looking at the solution, we can read off the roots of the characteristic equation. Indeed, the exponential shows that the real part of the root is 3 while the trigonometric functions show that the imaginary part is 1. Thus, a root of the characteristic equation is $r_1 = 3 + i$. The other root must be $r_2 = 3 - i$.

For an equation of the type

$$y'' + Ay' + By = 0$$

the characteristic equation becomes $r^2 + Ar + B = 0$ which has a root

$$-A/2 \pm \sqrt{A^2 - 4B}/2 = 3 + i.$$

Thus

$$-A/2 = 3, -\sqrt{A^2 - 4B}/2 = i$$

so that $A = -6$, $A^2 - 4B = -4$ hence $B = 10$. The differential equation is

$$y'' - 6y' + 10y = 0.$$

(ii) For the same differential equation, solve the initial value problem

$$y(0) = 1, \; y'(0) = 2.$$

Two solutions of the differential equation must be

$$y_1 = e^{3t}\cos t, y_2 = e^{3t}\sin t$$

hence the general solution is

$$y = e^{3t}(c_1\cos t + c_2\sin t).$$

We have $y(0) = c_1 = 1$ and

$$y'(t) = e^{3t}(-c_1\sin t + c_2\cos t) + 3e^{3t}(c_1\cos t + c_2\sin t) \implies y'(0) = c_2 + 3c_1 = 2 \implies c_2 = -1.$$

Hence

$$y = e^{3t}(\cos t - \sin t).$$

**Problem 5.**

Consider the differential equation

$$t^2y'' - 3ty' + 3y = 0, \text{ for } t > 0.$$

(i) Check that $y_1 = t$ and $y_2 = t^3$ form a fundamental pair of solutions.

To see that $y_1 = t$ is a solution we calculate

$$t^2y_1'' - 3ty_1' + 3y_1 = 0 - 3t + 3t = 0.$$

Similarly,

$$t^2y_2'' - 3ty_2' + 3y_2 = t^2 \cdot 6t - 3t \cdot 3t^2 + 3t^3 = 0.$$

Thus $y_2$ is a solution as well. We calculate

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = t \cdot (3t^2) - t^3 \cdot 1 = 2t^3 \neq 0.$$

Thus $y_1$ and $y_2$ are a fundamental pair of solutions.

(iii) What is the general solution to the differential equation?

We have

$$y = c_1y_1 + c_2y_2 = c_1t + c_2t^3.$$

**Problem 6.**

A tank originally contains 100 gallons of fresh water. Water containing 5 lb of salt per gallon is poured into the tank at a rate of $r$ gal/min. The mixture is allowed to leave the tank at the same rate. After 100 minutes there is exactly 50 lb of salt in the tank. What is the rate water was poured?

We write $Q(t)$ for the quantity of water. We must have $dQ/dt =$ rate in $-$ rate out. In our case,

$$\frac{dQ}{dt} = 5r - r\frac{Q}{100} = r \cdot \frac{500 - Q}{100}.$$

We separate variables to solve

$$\frac{dQ}{500 - Q} = \frac{r}{100}dt \implies -\ln|500-Q| = rt/100+C \implies 500-Q = Ce^{-rt/100} \implies Q(t) = 500-Ce^{-rt/100}.$$

Since $Q(0) = 0$ we must have $C = 500$ hence

$$Q(t) = 500(1 - e^{-rt/100}).$$

Since

$$Q(100) = 50 \implies 1 - e^{-r} = \frac{1}{10} \implies e^{-r} = \frac{9}{10} \implies r = \ln\frac{10}{9}.$$