Problem 1.

The system $\vec{x}' = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} \vec{x}$ has a repeated eigenvalue $\lambda = 4$ and an eigenvector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(i) Write down a fundamental pair of solutions.

One solution is given by

$$\vec{x}_1 = e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We find a generalized eigenvector

$$(A - 4I)\vec{w} = \vec{v} \implies \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \vec{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

A second solution is

$$\vec{x}_2 = e^{4t} \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

(ii) Calculate the matrix exponential e^{At}.

We have

$$\Psi(t) = e^{4t} \begin{bmatrix} 1 & t-1 \\ 1 & t \end{bmatrix}$$

Then

$$\Psi(0) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \implies \Psi(0)^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

We find

$$e^{At} = \Psi(t)\Psi(0)^{-1} = e^{4t} \begin{bmatrix} 1 & t-1 \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = e^{4t} \begin{bmatrix} -t+1 & t \\ -t & t+1 \end{bmatrix}.$$

Problem 2.

Using undetermined coefficients, find a particular solution to the differential equation:

$$y'' - 2y' - 3y = 3 - 10\sin t.$$

We look for a solution in the form

$$y = A\sin t + B\cos t + C.$$

Then

$$y' = A\cos t - B\sin t$$
$$y'' = -A\sin t - B\cos t$$

hence

$$y'' - 2y' - 3y = (-4A + 2B)\sin t + (-4B - 2A)\cos t - 3C = 10\sin t + 3.$$

Thus

$$-4B - 2A = 0,\ -4A + 2B = -10 \implies B = -1, A = 2$$

and $C = -1$ hence

$$y = -1 + 2\sin t - \cos t.$$

Problem 3.

Using variation of parameters, find a particular solution to the differential equation:

$$y'' - 2y' + 2y = e^t \sin t \cos t.$$

The homogeneous solutions are found by solving

$$r^2 - 2r + 2 = 0 \implies r = 1 \pm i.$$

We have

$$y_1 = e^t \cos t, \quad y_2 = e^t \sin t.$$

Clearly

$$W(y_1, y_2) = \begin{vmatrix} e^t \cos t & e^t \sin t \\ e^t(-\sin t + \cos t) & e^t(\cos t + \sin t) \end{vmatrix} = e^{2t}(\cos^2 t + \sin^2 t) = e^{2t}.$$

We have

$$u_1 = -\int \frac{e^t \cos t \sin t \cdot e^t \sin t}{e^{2t}}\, dt = -\int \sin^2 t \cos t \, dt = -\frac{1}{3}\sin^3 t$$

and

$$u_2 = \int \frac{e^t \cos t \sin t \cdot e^t \cos t}{e^{2t}}\, dt = \int \sin t \cos^2 t \, dt = -\frac{1}{3}\cos^3 t.$$

We find

$$y = -\frac{1}{3}e^t(\sin^3 t \cos t + \cos^3 t \sin t) = -\frac{1}{3}e^t \sin t \cos t(\sin^2 t + \cos^2 t) = -\frac{1}{3}e^t \sin t \cos t = -\frac{1}{6}e^t \sin 2t.$$

Problem 4.

Consider the system

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}.$$

(i) Write down the general solution.

We find the eigenvalues and eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 6 = 0 \implies \lambda_1 = 2, \lambda_2 = 3.$$

The eigenvector corresponding to $\lambda = 2$ is found via

$$(A - 2I)\vec{v}_1 = 0 \implies \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} v_1 = 0 \implies \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to $\lambda = -3$ is found via

$$(A + 3I)\vec{v}_2 = 0 \implies \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} v_2 = 0 \implies \vec{v}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

The general solution is

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

(ii) Sketch the trajectories.

The origin is a saddle.

(iii) Using variation of parameters, find a particular solution for the inhomogeneous system

$$\vec{x}' = A\vec{x} + \begin{bmatrix} 5e^t \\ 0 \end{bmatrix}.$$

We have

$$\Psi(t) = \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix}.$$

Thus

$$\Psi(t)^{-1} = \frac{-1}{5e^{-t}} \begin{bmatrix} -4e^{-3t} & -e^{-3t} \\ -e^{2t} & e^{2t} \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -4e^{-2t} & -e^{-2t} \\ -e^{3t} & e^{3t} \end{bmatrix}.$$

The particular solution is

$$x_p = \Psi(t) \int \Psi(t)^{-1} \begin{bmatrix} 5e^t \\ 0 \end{bmatrix} dt = \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix} \int \begin{bmatrix} 4e^{-t} \\ e^{4t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix} \begin{bmatrix} -4e^{-t} \\ \frac{1}{4}e^{4t} \end{bmatrix} = \begin{bmatrix} -\frac{15}{4}e^t \\ -5e^t \end{bmatrix}$$