Math 20D **Name:** ____________

October 22, 2018 **Midterm** **PID:** ____________

Instructions

1. **Write your *Name* and *PID* on the front of your Blue Book.**
2. **No calculators or other electronic devices are allowed during this exam.**
3. **You may use a double sided page of notes.**
4. **Write your solutions clearly in your Blue Book.**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order as they appear in the exam.**
   (c) **Start each numbered problem on a new side of a page.**
5. **Show all of your work and justify all your claims. No credit will be given for unsupported answers, even if correct.**

Complete 5 out of the 6 questions

1. (10 points) (a) Solve the initial value problem

$$y' - y = e^{-t}, \quad y(0) = y_0$$

(b) Explain how the behaviour of the solution $y(t)$ as $t \longrightarrow \infty$ depends on the value of the initial value y_0.

Hint: There are 3 different cases depending on y_0.

Solution. (a) This is a linear equation with:

$$P(t) = -1$$

Calculate $\mu(t)$

$$\mu(t) = e^{\int P(t)dt} = e^{\int -1\ dt} = e^{-t}$$

Multiply by $\mu(t)$ to get:

$$e^{-t}y' - e^{-t}y = e^{-2t}$$

LHS simplifies with the product rule:

$$\frac{d}{dt}\left(y(t) \cdot e^{-t}\right) = e^{-2t}$$

Integrate to get:

$$y(t) \cdot e^{-t} = -\frac{1}{2}e^{-2t} + C$$

Divide by $\mu(t)$ to get:

$$y(t) = -\frac{1}{2}e^{-t} + Ce^{t}$$

Substitute $t = 0$ to apply the initial condition $y(0) = y_0$

$$y_0 = -\frac{1}{2} + C \implies C = y_0 + \frac{1}{2}$$

An explicit solution is then given by:

$$y(t) = -\frac{1}{2}e^{-t} + (y_0 + \frac{1}{2})e^t$$

(b) First notice that the function $-\frac{1}{2}e^{-t} \longrightarrow 0$ as $t \longrightarrow \infty$ for any value of y_0.
So the behaviour of $y(t)$ for large t depends only on $(y_0+\frac{1}{2})e^t$, in particular it is determined by the sign of $(y_0 + \frac{1}{2})$:

Case 1 If $y_0 > -\frac{1}{2}$ then $(y_0 + \frac{1}{2})$ is positive and so $y(t) \longrightarrow +\infty$ as $t \longrightarrow \infty$

Case 2 If $y_0 = -\frac{1}{2}$ then $(y_0 + \frac{1}{2})$ is 0 and so $y(t)$ approaches 0 as $t \longrightarrow \infty$

Case 3 If $y_0 < -\frac{1}{2}$ then $(y_0 + \frac{1}{2})$ is negative and so $y(t)$ diverges to $-\infty$ as $t \longrightarrow \infty$

□

2. (10 points) Find the general solution to the following differential equations

(a)
$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$$

(b)
$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 3y = 0$$

(c)
$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0$$

Solution. (a) The auxiliary equation is

$$r^2 + 6r + 9 = 0$$

with repeated root $r_0 = -3$, the general solution is then given by:

$$y(t) = c_1e^{-3t} + c_2te^{-3t}$$

(b) The auxiliary equation is

$$4r^2 + 4r - 3 = 0$$

with distinct roots $r_1 = -\frac{3}{2}$, $r_2 = \frac{1}{2}$, the general solution is then given by:

$$y(t) = c_1e^{-\frac{3}{2}t} + c_2e^{\frac{1}{2}t}$$

(c) The auxiliary equation is

$$r^2 - 2r + 5 = 0$$

with complex conjugate roots $r_1 = 1 + 2i$, $r_2 = 1 - 2i$, the general solution is then given by:

$$y(t) = c_1e^t\cos(2t) + c_2e^t\sin(2t)$$

□

3. (10 points) (a) Find the explicit solution to the initial value problem

$$e^t - yy' = 0, \quad y(0) = 1$$

(b) For which values of t is this solution valid?

Solution. (a) This is definitely not a linear equation, so we need to check whether it is separable or exact.

It is easier to check whether it is separable by rewriting y' as $\frac{dy}{dt}$ and rearranging to get:

$$\frac{dy}{dt} = \frac{e^t}{y} = \left(e^t\right) \cdot \left(\frac{1}{y}\right)$$

This is a separable equation, multiplying by dt and dividing by $\frac{1}{y}$ we get:

$$ydy = e^t dt$$

Integrate:

$$\frac{1}{2}y^2 = e^t + C$$

Substitute $t = 0$ and apply initial condition:

$$\frac{1}{2} = 1 + C \implies C = -\frac{1}{2}$$

So an implicit solution is given by:

$$\frac{1}{2}y^2 = e^t - \frac{1}{2}$$

Rearranging and taking a square root we get:

$$y(t) = \pm\sqrt{2e^t - 1}$$

Since we must have $y(0) = 1$ we are forced to choose the positive square root:

$$y(t) = \sqrt{2e^t - 1}$$

(b) $y(t)$ will only exist for values of t such that $2e^t - 1 \geq 0$. Simplifying further:

$$2e^t - 1 \geq 0 \implies e^t \geq \frac{1}{2} \implies t \geq \ln\left(\frac{1}{2}\right) \implies t \geq -\ln(2)$$

□

4. (10 points) Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{x - y - 1}{x + y + 5}$$

Solution. Simplify by multiplying by $(x+y+5)$ and moving everything to the LHS

$$(x+y+5)\frac{dy}{dx} - (x-y-1) = 0$$

Compare with the form $M + N\frac{dy}{dx} = 0$ and identify $M(x,y)$ and $N(x,y)$:

$$M(x,y) = -x+y+1$$
$$N(x,y) = x+y+5$$

Check for exactness:

$$M_y = 1$$
$$N_x = 1$$

$M_y = N_x \implies$ Exact equation

Integrate M with respect to x and N with respect to y and compare:

$$\int M dx = \int (-x+y+1)dx = -\frac{1}{2}x^2 + xy + x$$
$$\int N dy = \int (x+y+5)dy = xy + \frac{1}{2}y^2 + 5y$$

Collect together the unique terms to get:

$$F(x,y) = -\frac{1}{2}x^2 + xy + x + \frac{1}{2}y^2 + 5y$$

The general solution is then given by the level curves $F(x,y) = C$ where C is an arbitrary constant:

$$-\frac{1}{2}x^2 + xy + x + \frac{1}{2}y^2 + 5y = C$$

□

5. (10 points) Consider the following nonhomogeneous equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = f(t)$$

Determine the form of a particular solution in the following cases (**you do not have to calculate any unknown coefficients**)

(a) $f(t) = t^2 - 1$

(b) $f(t) = e^{2t}$

(c) $f(t) = \cos(2t)$

(d) $f(t) = te^t$

Solution. Note the roots of the auxiliary equation $r^2 + r - 2 = 0$ are $r_1 = -2$ and $r_2 = 1$

(a) $y_p = At^2 + Bt + C$

(b) $y_p = Ae^{2t}$ (no extra t because 2 is not an auxiliary root)

(c) $y_p = A\cos(2t) + B\sin(2t)$

(d) $y_p = (At + B)te^t$ (extra t here because 1 is a root to the auxiliary equation)

□

6. (10 points) Solve the initial value problem

$$4\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + x = 0; \quad x(0) = 2, \quad x'(0) = 2$$

Solution. Write down the auxiliary equation and solve:

$$4r^2 - 4r + 1 \implies (2r - 1)^2 = 0 \implies r_0 = \frac{1}{2} \text{ (\textbf{repeated root})}$$

The general solution is then given by:

$$x(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$$

To apply the second initial condition we need to differentiate $x(t)$:

$$x'(t) = \frac{1}{2}c_1 e^{\frac{1}{2}t} + c_2\left(e^{\frac{1}{2}t} + \frac{1}{2}te^{\frac{1}{2}t}\right)$$

Substitute $t = 0$ and apply initial conditions:

$$\begin{aligned} 2 &= c_1 \\ 2 &= \frac{1}{2}c_1 + c_2 \end{aligned}$$

Solving we get $c_1 = 2$, $c_2 = 1$, the solution to this IVP is then given by:

$$x(t) = 2e^{\frac{1}{2}t} + te^{\frac{1}{2}t}$$

□