Math 20D **Name:** ____________

November 19, 2018 **Midterm 2** **PID:** ____________

Instructions

1. **Write your *Name* and *PID* on the front of your Blue Book.**
2. **No calculators or other electronic devices are allowed during this exam.**
3. **You may use a double sided page of notes.**
4. **Write your solutions clearly in your Blue Book.**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order as they appear in the exam.**
   (c) **Start each numbered problem on a new side of a page.**
5. **Show all of your work and justify all your claims. No credit will be given for unsupported answers, even if correct.**

Complete 5 out of the 6 questions

1. (10 points) Solve the initial value problem

$$x^2y''(x) = 5xy'(x) - 13y(x), \quad y(-1) = 1,\ y'(-1) = -1 \tag{1}$$

Solution. Rearrange (1) into standard form:

$$x^2y''(x) = 5xy'(x) - 13y(x) \implies x^2y''(x) - 5xy'(x) + 13y(x) = 0$$

The characteristic equation of a Cauchy-Euler equation $at^2y'' + bty' + cy = 0$ is given by $ar^2 + (b-a)r + c = 0$, so in this case:

$$r^2 - 6r + 13 = 0$$

Complex conjugate solutions $r = 3 \pm 2i$

Solutions to Cauchy-Euler equations are either valid for $t > 0$ or for $t < 0$

The general solution to a Cauchy-Euler equation with complex conjugate roots $\alpha \pm i\beta$ is given by:

$$y(t) = c_1t^\alpha \cos(\beta \ln t) + c_2t^\alpha \sin(\beta \ln t) \quad \textbf{for t} > \mathbf{0}$$

and by

$$y(t) = c_1(-t)^\alpha \cos(\beta \ln(-t)) + c_2(-t)^\alpha \sin(\beta \ln(-t)) \quad \textbf{for t} < \mathbf{0}$$

In our case $\alpha = 3$, $\beta = 2$, and initial conditions are given at $t = -1 < 0$, so $y(t)$ has the form:

$$\begin{aligned} y(t) &= c_1(-t)^3 \cos(2\ln(-t)) + c_2(-t)^3 \sin(2\ln(-t)) \\ &= d_1t^3 \cos(2\ln(-t)) + d_2t^3 \sin(2\ln(-t)) \end{aligned} \tag{2}$$

(renamed $d_1 = -c_1$ and $d_2 = -c_2$ above)

We determine the constants d_1 and d_2 by applying the initial conditions. First (to use $y'(-1) = -1$) we will need to differentiate $y(t)$

Application of the chain rule gives

$$\begin{aligned}\frac{d}{dt}\left(\cos(2\ln(-t))\right) &= -\sin(2\ln(-t))\cdot\frac{2}{-t}\cdot -1 = -\frac{2\sin(2\ln(-t))}{t}\\ \frac{d}{dt}\left(\sin(2\ln(-t))\right) &= \cos(2\ln(-t))\cdot\frac{2}{-t}\cdot -1 = \frac{2\cos(2\ln(-t))}{t}\end{aligned}$$

In particular

$$\begin{aligned}y'(t) &= d_1\left(3t^2\cos(2\ln(-t)) + t^3\left(-\frac{2\sin(2\ln(-t))}{t}\right)\right)\\ &\quad + d_2\left(3t^2\sin(2\ln(-t)) + t^3\left(\frac{2\cos(2\ln(-t))}{t}\right)\right)\\ &= d_1t^2\left(3\cos(2\ln(-t)) - 2\sin(2\ln(-t))\right) d_2t^2\left(3\sin(2\ln(-t)) + 2\cos(2\ln(-t))\right)\end{aligned} \tag{3}$$

Substituting $t = -1$ into (2) and (3) and applying the initial conditions we get:

$$\begin{aligned}1 &= y(-1) = -d_1\cos(2\ln(1)) - d_2\sin(2\ln(1))\\ -1 &= y'(-1) = d_1\left(3\cos(2\ln(1)) - 2\sin(2\ln(1))\right) + d_2\left(3\sin(2\ln(1)) + 2\cos(2\ln(1))\right)\end{aligned}$$

Simplifying:

$$\begin{aligned}1 &= -d_1 \implies d_1 = -1\\ -1 &= 3d_1 + 2d_2 \implies d_2 = 1\end{aligned}$$

So the solution to (1) is:

$$y(t) = -t^3\cos(2\ln(-t)) + t^3\sin(2\ln(-t))$$

□

2. (10 points) Find the general solution to the following differential equation

$$y'' + 16y = \tan(4t) \tag{4}$$

Solution. Notice that we cannot apply the method of undetermined coefficients to $g(t) = \tan(4t)$, so we proceed by variation of parameters.

First find two linearly independent homogeneous solutions.

Auxiliary equation:

$$r^2 + 16 = 0 \implies r = \pm 4i$$

Two linearly independent solutions:

$$\begin{aligned}y_1 &= \cos(4t)\\ y_2 &= \sin(4t)\end{aligned}$$

Given a differential equation $ay'' + by' + cy = g$ with homogeneous solutions y_1 and y_2 a particular solution is given by:

$$y_p(t) = y_1(t)v_1(t) + y_2(t)v_2(t) \tag{5}$$

where v_1 and v_2 can be calculated using the formulae:

$$v_1(t) = \int \frac{-gy_2}{aW[y_1, y_2](t)} dt$$
$$v_2(t) = \int \frac{gy_1}{aW[y_1, y_2](t)} dt$$

The function on the denominator is the Wronskian of $y_1(t)$ and $y_2(t)$

$$W[y_1, y_2](t) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

In our case:

$$W[y_1, y_2](t) = \det \begin{pmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{pmatrix} = 4$$

And so:

$$v_1(t) = -\frac{1}{4}\int \tan(4t)\sin(4t)dt = -\frac{1}{4}\int \frac{\sin^2(4t)}{\cos(4t)}dt$$
$$v_2(t) = -\frac{1}{4}\int \tan(4t)\cos(4t)dt = \frac{1}{4}\int \sin(4t)dt$$

$v_1(t)$ is evaluated by rewriting the numerator $\sin^2(4t) = 1 - \cos^2(4t)$

$$\begin{aligned} v_1(t) &= -\frac{1}{4}\int \frac{1-\cos^2(4t)}{\cos(4t)}dt \\ &= -\frac{1}{4}\int \sec(4t) - \cos(4t)dt \\ &= -\frac{1}{4}\left(\frac{1}{4}\ln|\sec(4t)+\tan(4t)| - \frac{1}{4}\sin(4t)\right) \\ &= -\frac{1}{16}\ln|\sec(4t)+\tan(4t)| + \frac{1}{16}\sin(4t) \end{aligned}$$

$v_2(t)$ is easier

$$v_2(t) = \frac{1}{4}\int \sin(4t)dt = -\frac{1}{16}\cos(4t)$$

Substituting y_1, y_2, v_1, v_2 into (5) we get:

$$\begin{aligned} y_p &= \left(-\frac{1}{16}\ln|\sec(4t)+\tan(4t)| + \frac{1}{16}\sin(4t)\right)\cos(4t) + \left(-\frac{1}{16}\cos(4t)\right)\sin(4t) \\ &= -\frac{1}{16}\cos(4t)\ln|\sec(4t)+\tan(4t)| \end{aligned}$$

The general solution to (4) is then given by:

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= c_1\cos(4t) + c_2\sin(4t) - \frac{1}{16}\cos(4t)\ln|\sec(4t)+\tan(4t)| \end{aligned}$$

□

3. (10 points) Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \tag{6}$$

Solution. To solve a system of differential equations $\mathbf{x}' = A\mathbf{x}$ we calculate the eigenvalues and eigenvectors of A.

Eigenvalues are solutions to the characteristic equation given by:

$$\det(A - rI) = 0$$

In our case this is

$$\det \begin{pmatrix} 1-r & 1 & 0 \\ 0 & 2-r & 0 \\ 0 & -1 & 4-r \end{pmatrix} = (r-1)(r-2)(r-4) = 0 \implies r = 1, 2, 4$$

Calculating eigenvector $\mathbf{u_1}$ associated to eigenvector $r_1 = 1$

We determine $\mathbf{u_1}$ by solving the matrix equation $(A - r_1 I)\mathbf{u_1} = \mathbf{0}$, let $\mathbf{u_1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

We need to solve:

$$\begin{pmatrix} 1-(1) & 1 & 0 \\ 0 & 2-(1) & 0 \\ 0 & -1 & 4-(1) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Simplifying this is the matrix equation:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Or equivalently the simultaneous equations:

$$\begin{aligned} y &= 0 \\ y &= 0 \\ -y + 3z &= 0 \end{aligned}$$

Solving these we get $y = z = 0$, notice that there are no conditions on x.

This means:

$$\mathbf{u_1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Any choice of x will give us an eigenvector (except $x = 0$), letting $x = 1$ we get the eigenvector

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Similarly one calculates eigenvectors $\mathbf{u_2}$, $\mathbf{u_3}$ associated to eigenvalues $r_2 = 2$, $r_3 = 2$ respectively. You should get:

$$\mathbf{u_2} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{u_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The general solution to (6) is then given by:

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Equivalently this can be written as:

$$\mathbf{x}(t) = \begin{pmatrix} e^t & 2e^{2t} & 0 \\ 0 & 2e^{2t} & 0 \\ 0 & e^{2t} & e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Subtitute $t = 0$ and apply the initial condition:

$$\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} e^0 & 2e^0 & 0 \\ 0 & 2e^0 & 0 \\ 0 & e^0 & e^0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Solving

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

we get

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

The solution to (6) is then given by:

$$\mathbf{x}(t) = \begin{pmatrix} e^t & 2e^{2t} & 0 \\ 0 & 2e^{2t} & 0 \\ 0 & e^{2t} & e^{4t} \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -e^t - 2e^{2t} \\ -2e^{2t} \\ -e^{2t} + 2e^{4t} \end{pmatrix}$$

□

4. (10 points) Find the general solution to the non-homogeneous system

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -6e^{-t} \\ -4e^{-t} \end{pmatrix} \tag{7}$$

Solution. This will have two parts.

Calculating homogeneous solution

Proceed as in Question 3 and solve the characteristic equation

$$\det \begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix} = 0$$

which simplifies to $r^2 - 2r + 5 = 0$. By completing the square or otherwise we get complex conjugate solutions $r = 1 \pm 2i$.

A system of differential equations $\mathbf{x}' = A\mathbf{x}$ with complex eigenvalues $\alpha \pm \beta$ and corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$ has two linearly independent homogeneous solutions:

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\alpha t} \left(\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t) \right) \\ \mathbf{x}_2(t) &= e^{\alpha t} \left(\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t) \right) \end{aligned}$$

The general (homogeneous) solution is then given by:

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) \\ &= c_1 e^{\alpha t} \left(\mathbf{a} \cos(\beta t) - \mathbf{b} \sin(\beta t) \right) + c_2 e^{\alpha t} \left(\mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t) \right) \end{aligned} \tag{8}$$

We need to determine the complex eigenvector $\mathbf{u} = \mathbf{a} + i\mathbf{b}$ associated to the complex eigenvalue $\alpha + i\beta = 1 + 2i$ by solving

$$(A - (1+2i)I)\mathbf{u} = \mathbf{0}$$

This is the matrix equation:

$$\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which simplifies to

$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Or equivalently:

$$\begin{aligned} (2-2i)x - 2y &= 0 \\ 4x + (-2-2i)y &= 0 \end{aligned}$$

Notice that the second equation here is a scalar multiple of the first (multiply the first line by $1+i$), so we can ignore the second equation (since it won't give us any new information).

Rearranging the first equation:

$$y = (1-i)x$$

In particular:

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ (1-i)x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

Again any choice of x (except 0) gives an eigenvector. Let $x = 1$ and separate the real and imaginary components:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \mathbf{a} + i\mathbf{b}$$

So

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Substituting $\alpha = 1, \beta = 2, \mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ into (8) we get the general homogeneous solution:

$$\begin{aligned} \mathbf{x}_h(t) &= c_1 e^t \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin(2t) \right) + c_2 e^t \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(2t) \right) \\ &= c_1 e^t \begin{pmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix} \end{aligned}$$

Finding a particular solution

Applying the method of undetermined coefficients, we make a guess for the particular solution:

$$\mathbf{x}_p(t) = \mathbf{a} e^{-t}$$

The idea is to substitute this guess into (7).

First we will need to differentiate:

$$\mathbf{x}_p'(t) = -\mathbf{a} e^{-t}$$

Now substituting $\mathbf{x}_p(t)$ and $\mathbf{x}_p'(t)$ into (7):

$$-\mathbf{a} e^{-t} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{a} e^{-t} + \begin{pmatrix} -6e^{-t} \\ -4e^{-t} \end{pmatrix}$$

Cancelling e^{-t}:

$$-\mathbf{a} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{a} + \begin{pmatrix} -6 \\ -4 \end{pmatrix} \implies \begin{pmatrix} 4 & -2 \\ 4 & 0 \end{pmatrix} \mathbf{a} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Solving we get:

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore a particular solution to (7) is given by:

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

And so the general solution to (7) is:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_h(t) + \mathbf{x}_p(t) \\ &= c_1 e^t \begin{pmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix} \end{aligned}$$

□

5. (10 points) (a) Verify that

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} \right\}$$

is a fundamental solution set to the homogeneous system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} \tag{9}$$

(b) Using variation of parameters, find a particular solution to the non-homogeneous system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

Solution. (a) Let $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

To verify that $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} \right\}$ is a fundamental solution set we need to check that:

(1) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$ are solutions to (9)

(2) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$ are linearly independent.

We can check both simultaneously by forming a matrix X with columns $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

$$X = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

and verifying:

(1) $X' = AX$ (this shows the columns of X are solutions)

(2) $\det(X) = 0$ (this shows that the columns of X are linearly independent)

Checking (1)

Calculate:

$$X' = \begin{pmatrix} e^t & -e^{-t} \\ e^t & -3e^{-t} \end{pmatrix}$$

$$AX = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} = \begin{pmatrix} e^t & -e^{-t} \\ e^t & -3e^{-t} \end{pmatrix}$$

Checking (2)

$$\det(X) = \det \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} = 3 - 1 = 2 \neq 0$$

(b) We showed in part (a) that X is a fundamental matrix for the system (9).

Variation of parameters is a method for calculating a particular solution $\mathbf{x}_p$ to a non-homogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$

$$\mathbf{x}_p = X\mathbf{v}$$

where $\mathbf{v}$ is calculated with the formula:

$$\mathbf{v} = \int X^{-1}\mathbf{f}$$

Here X^{-1} is the inverse matrix to X

$$X^{-1} = \frac{1}{\det X}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{pmatrix}$$

Calculate

$$X^{-1}\mathbf{f} = \frac{1}{2}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{pmatrix}\begin{pmatrix} e^{t} \\ t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 - te^{-t} \\ -e^{2t} + te^{t} \end{pmatrix}$$

Integrate to get $\mathbf{v}$:

$$\mathbf{v} = \int \frac{1}{2}\begin{pmatrix} 3 - te^{-t} \\ -e^{2t} + te^{t} \end{pmatrix}$$

To deal with both $\int te^{\pm t}$ together you can calculate:

$$\int te^{\lambda t} = \frac{te^{\lambda t}}{\lambda} - \frac{1}{\lambda^2}e^{\lambda t}$$

Applying above formula for $\lambda = 1, -1$

$$\int te^{t} = te^{t} - e^{t}$$
$$\int te^{-t} = -te^{-t} - e^{-t}$$

So

$$\begin{aligned} \mathbf{v} &= \int \begin{pmatrix} \frac{3}{2} - \frac{1}{2}te^{-t} \\ -\frac{1}{2}e^{2t} + \frac{1}{2}te^{t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2}t - \frac{1}{2}(-te^{-t} - e^{-t}) \\ -\frac{1}{4}e^{2t} + \frac{1}{2}(te^{t} - e^{t}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2}t + \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t} \\ -\frac{1}{4}e^{2t} + \frac{1}{2}te^{t} - \frac{1}{2}e^{t} \end{pmatrix} \end{aligned}$$

Finally

$$\begin{aligned} \mathbf{x}_p &= X\mathbf{v} \\ &= \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\begin{pmatrix} \frac{3}{2}t + \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t} \\ -\frac{1}{4}e^{2t} + \frac{1}{2}te^{t} - \frac{1}{2}e^{t} \end{pmatrix} \end{aligned}$$

Separate calculation into components for clarity

The first component $\mathbf{x}_p$ is:

$$= e^t\left(\frac{3}{2}t + \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t}\right) + e^{-t}\left(-\frac{1}{4}e^{2t} + \frac{1}{2}te^t - \frac{1}{2}e^t\right)$$
$$= \frac{3}{2}te^t - \frac{1}{4}e^t + t$$

The second component $\mathbf{x}_p$ is:

$$= e^t\left(\frac{3}{2}t + \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t}\right) + 3e^{-t}\left(-\frac{1}{4}e^{2t} + \frac{1}{2}te^t - \frac{1}{2}e^t\right)$$
$$= \frac{3}{2}te^t - \frac{3}{4}e^t + 2t - 1$$

So finally

$$\mathbf{x}_p = \frac{1}{4}\begin{pmatrix} 6te^t - e^t + 4t \\ 6te^t - 3e^t + 8t - 4 \end{pmatrix}$$

□

6. (10 points) Find a fundamental matrix for the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{x} \tag{10}$$

Solution. It is a general fact that a fundamental matrix to any system of differential equations $\mathbf{x}' = A\mathbf{x}$ is given by the matrix e^{At}. By definition:

$$e^{At} := I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

In general it is hard to calculate e^{At} (this is why we normally find generalized eigenvalues $\mathbf{u}$ and calculate $e^{At}\mathbf{u}$ with a different formula!).

However here if one notices that the coefficient matrix above (which we will call A) is nilpotent, meaning $A^k = 0$ for some k, then it is actually possible to calculate e^{At} directly.

Observe:

$$A = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$
$$A^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $A^3 = 0$, and therefore $A^4, A^5, \cdots$ and all higher powers are also all 0. In particular:

$$\begin{aligned}
e^{At} &= I + At + A^2\frac{t^2}{2!} + (0 + \cdots) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{t^2}{2} \\
&= \begin{pmatrix} 1 & t & 4t + t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

This is a fundamental matrix to (10). □