Math 20D - Spring 2017 - Final Exam

Problem 1.

Using undetermined coefficients, find the general solution of the differential equation

$$y'' - 2y' = e^{2t} - 4t.$$

`Solution:` *The homogeneous equation* $y'' - 2y' = 0$ *has characteristic equation* $r^2 - 2r = 0$ *which gives* $r = 0$ *and* $r = 2$. *Thus, a fundamental pair of solutions for the homogeneous equation is given by*

$$y_1 = e^{0\cdot t} = 1, \;\; y_2 = e^{2t}.$$

The homogeneous solution is

$$y_h = c_1 + c_2 e^{2t}.$$

For the particular solution, we seek

$$y_p = Ate^{2t} + (Bt^2 + Ct + D).$$

The presence of te^{2t} *is motivated by the fact that we need not replicate any of the homogeneous solutions. Similarly, the degree of the polynomial part of the solution is seen to be* 2 *because of the presence of* y' *and the term* t *in the answer. We have*

$$y_p' = A(2t+1)e^{2t} + (2Bt + C)$$
$$y_p'' = A(4t+4)e^{2t} + 2B.$$

Therefore

$$y_p'' - 2y_p' = \big(A(4t+4)e^{2t} + 2B\big) - 2\left((2t+1)e^{2t} + (2Bt+C)\right) = 2Ae^{2t} + (-4Bt + 2B - C) = e^{2t} - 4t.$$

This gives

$$2A = 1, -4B = -4, 2B - 2C = 0.$$

Thus

$$A = \frac{1}{2}, B = 1, C = 1 \implies y_p = \frac{1}{2}te^{2t} + t^2 + t.$$

We chose here $D = 0$ *since we need only one particular solution. The general solution is found by superimposing*

$$y = y_p + c_1 y_1 + c_2 y_2 = \frac{1}{2}te^{2t} + t^2 + t + c_1 + c_2 e^{2t}.$$

Problem 2.

Using integrating factors, find the general solution of the differential equation

$$ty' = t\cos t^4 - 3y.$$

`Solution:` *We first write the equation in standard form*

$$ty' = t\cos t^4 - 3y \implies ty' + 3y = t\cos t^4 \implies y' + \frac{3}{t}y = \cos t^4.$$

The integrating factor is

$$u = \exp\left(\int \frac{3}{t}\,dt\right) = \exp(3\ln t) = t^3.$$

Multiplying by the integrating factor throughout we find

$$(t^3y)' = t^3\cos t^4 \implies t^3y = \int t^3\cos t^4\,dt = \frac{1}{4}\sin t^4 + C.$$

This gives

$$y = \frac{1}{4t^3}\sin t^4 + \frac{C}{t^3}.$$

Problem 3.

Consider the differential equation

$$x^2y'' - 2xy' + (2 - x^2)y = x^3e^x.$$

(i) Find the values of r for which $y = xe^{rx}$ is a solution to the *homogeneous* equation.
(ii) Using variation of parameters, find a particular solution to the *inhomogeneous* equation.

`Solution:`

(i) *If $y = xe^{rx}$ then direct computation shows*

$$y' = (rx + 1)e^{rx}, \;\; y'' = (r^2x + 2r)e^{rx}.$$

Thus

$$x^2y'' - 2xy' + (2 - x^2)y = e^{rx} \cdot \left(x^2(r^2x + 2r) - 2x(rx + 1) + (2 - x^2)x\right) = e^{rx}(r^2x^3 - x^3) = e^{rx}x^3(r^2 - 1).$$

For the homogeneous equation, the last expression should be 0 *for all* x*, hence* $r^2 - 1 = 0$ *so* $r = \pm 1$*. The two solutions are*

$$y_1 = xe^x, y_2 = xe^{-x}.$$

(ii) *We look for a particular solution*

$$y_p = u_1y_1 + u_2y_2.$$

First, we bring the equation into standard form

$$y'' - \frac{2}{x} \cdot y' + \frac{2 - x^2}{x^2} \cdot y = xe^x.$$

We have

$$W(y_1, y_2) = \begin{vmatrix} xe^x & xe^{-x} \\ (x+1)e^x & (-x+1)e^{-x} \end{vmatrix} = xe^x \cdot (-x + 1)e^{-x} - xe^x \cdot (x + 1)e^{-x} = -2x^2.$$

By variation of parameters, we have

$$u_1 = -\int \frac{xe^x}{-2x^2} \cdot (xe^{-x})\, dx = \int \frac{1}{2}\, dx = \frac{x}{2},$$

$$u_2 = \int \frac{xe^x}{-2x^2} \cdot (xe^x)\, dx = \int \frac{-1}{2} e^{2x}\, dx = -\frac{1}{4}e^{2x}.$$

Then

$$y_p = \frac{x}{2} \cdot (xe^x) - \frac{1}{4}e^{2x} \cdot (xe^{-x}) = \frac{x^2}{2}e^x - \frac{1}{4}xe^x.$$

Problem 4.

Consider the system $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}.$$

(i) Find a fundamental pair of solutions to the system.

(ii) Draw the trajectories of the general solution. What is the type of the phase portrait you obtained?

(iii) Calculate the normalized fundamental matrix $\Phi(t)$ with $\Phi(0) = I$.

(iv) Solve the initial value problem $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(v) Use variation of parameters to find a particular solution the following inhomogeneous system

$$\vec{x}' = Ax + \begin{bmatrix} te^{3t} \\ 0 \end{bmatrix}.$$

`Solution:`

(i) *We find Tr* $A = 6, \det A = 9$. *The eigenvalues are roots of the characteristic polynomial*

$$\lambda^2 - 6\lambda + 9 = 0 \implies \lambda = 3.$$

This is a repeated eigenvalue and the matrix is defective. We find the eigenvector by computing

$$A - 3I = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}.$$

Thus

$$(A - 3I)\vec{v} = 0 \implies \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus

$$\vec{x}_1 = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We find a generalized eigenvector by solving

$$(A - 3I)\vec{w} = \vec{v} \implies \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \vec{w} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}.$$

Other choices for $\vec{v}, \vec{w}$ *are possible here. We have*

$$\vec{x}_2 = e^{3t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right).$$

(ii) *The general solution is found by superimposing the two solutions found above*

$$\vec{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right).$$

The trajectory is an improper node source. The dominant term is $e^{3t} t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and solutions follow the direction* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *both when* $t \to -\infty$ *and when* $t \to \infty$. *To determine the direction of the trajectory, we need to compute the velocity vector at one point. For instance, we can pick*

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \vec{x}'(0) = A\vec{x}(0) = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

This vector points down. Since the trajectories diverge away from the origin, in order to match the direction of the velocity vector, the trajectories must move clockwise.

(iii) *We have*

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{3t}(t-1/2) \\ e^{3t} & e^{3t}t \end{bmatrix}.$$

Note that

$$\Psi(0) = \begin{bmatrix} 1 & -1/2 \\ 1 & 0 \end{bmatrix} \implies \Psi(0)^{-1} = \frac{1}{1/2}\begin{bmatrix} 0 & 1/2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}.$$

Thus

$$\Phi(t) = \Psi(t)\cdot\Psi(0)^{-1} = \begin{bmatrix} e^{3t} & e^{3t}(t-1/2) \\ e^{3t} & e^{3t}t \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} (1-2t)e^{3t} & 2te^{3t} \\ -2te^{3t} & (1+2t)e^{3t} \end{bmatrix}.$$

(iv) *We have*

$$\vec{x} = \Phi(t)\vec{x}(0) = \begin{bmatrix} (1-2t)e^{3t} & 2te^{3t} \\ -2te^{3t} & (1+2t)e^{3t} \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (1-2t)e^{3t} \\ -2te^{3t} \end{bmatrix}.$$

(v) *We have*

$$\vec{x}_p = \Psi(t)\int \Psi(t)^{-1}\begin{bmatrix} te^{3t} \\ 0 \end{bmatrix} dt.$$

We compute $\det\Psi(t) = e^{6t}/2$ *so that*

$$\Psi(t)^{-1} = \frac{1}{e^{6t}/2}\begin{bmatrix} te^{3t} & -e^{3t}(t-1/2) \\ -e^{3t} & e^{3t} \end{bmatrix}.$$

Then

$$\Psi(t)^{-1}\begin{bmatrix} t^2e^{3t} \\ 0 \end{bmatrix} = \frac{1}{e^{6t}/2}\begin{bmatrix} te^{3t} & -e^{3t}(t-1/2) \\ -e^{3t} & e^{3t} \end{bmatrix}\begin{bmatrix} te^{3t} \\ 0 \end{bmatrix} = \frac{2}{e^{6t}}\begin{bmatrix} t^2e^{6t} \\ -te^{6t} \end{bmatrix} = \begin{bmatrix} 2t^2 \\ -2t \end{bmatrix}.$$

Substituting, we obtain

$$\vec{x}_p = \begin{bmatrix} e^{3t} & e^{3t}(t-1/2) \\ e^{3t} & e^{3t}t \end{bmatrix}\int\begin{bmatrix} 2t^2 \\ -2t \end{bmatrix} dt = \begin{bmatrix} e^{3t} & e^{3t}(t-1/2) \\ e^{3t} & e^{3t}t \end{bmatrix}\begin{bmatrix} 2t^3/3 \\ -t^2 \end{bmatrix} = e^{3t}\begin{bmatrix} 2t^3/3 - t^2(t-1/2) \\ 2t^3/3 - t^2\cdot t \end{bmatrix}.$$

Thus

$$\vec{x}_p = e^{3t}\begin{bmatrix} -t^3/3 + t^2/2 \\ -t^3/3 \end{bmatrix}.$$

Problem 5.

Consider the differential equation

$$y'' - 3xy' - 3y = 0 \text{ with initial conditions } y(0) = 1, y'(0) = 0$$

whose solution is written as a power series

$$y = a_0 + a_1 x + a_2 x^2 + \ldots.$$

(i) Using the initial conditions, calculate the coefficients a_0 and a_1.

(ii) Find the recurrence relation between the coefficients of the power series y.

(iii) Write down the first four *non-zero* terms of the solution. Is the solution even or odd?

(iv) Write down the general expression for the non-zero coefficients. Express the solution y in closed form. The final answer should be a familiar function. You may need to recall the series expansion

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots + \frac{w^n}{n!} + \ldots.$$

`Solution:`

(i) *Substituting* $x = 0$ *we obtain*

$$y(0) = a_0 = 1$$

and computing derivatives we find

$$y'(0) = a_1 = 0.$$

Thus $a_0 = 1, a_1 = 0$.

(ii) *We have* $y = \sum_{n=0}^{\infty} a_n x^n$ *which gives*

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \implies xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

where we reinserted the term $n = 0$ *since the expression above covers this case as well. Next,*

$$y'' = \sum_{n=2} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n.$$

Thus

$$y''-3xy'-3y = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - 3\sum_{n=0}^{\infty} n a_n x^n - 3\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left((n+1)(n+2)a_{n+2} - (3n+3)a_n\right) x^n = 0.$$

Therefore

$$(n+1)(n+2)a_{n+2} - (3n+3)a_n = 0 \implies a_{n+2} = \frac{3(n+1)}{(n+1)(n+2)} a_n \implies a_{n+2} = \frac{3}{n+2} a_n.$$

(iii) *We have* $a_1 = 0$. *The above recursions works in steps of* 2, *so* $a_n = 0$ *for all* n *odd. Thus the solution only has even terms, hence* y *is even.*

We use the recurrence for $n = 0, 2, 4, 6$ *to find*

$$a_0 = 1, \quad a_2 = \frac{3}{2} a_0 = \frac{3}{2}$$

$$a_4 = \frac{3}{4}a_2 \implies a_4 = \frac{3}{4}\cdot\frac{3}{2}$$
$$a_6 = \frac{3}{6}\cdot a_4 \implies a_6 = \frac{3}{6}\cdot\frac{3}{4}\cdot\frac{3}{2}.$$

Thus

$$y = 1 + \frac{3}{2}x^2 + \frac{3}{4}\cdot\frac{3}{2}x^4 + \frac{3}{6}\cdot\frac{3}{4}\cdot\frac{3}{2}x^6 + \dots.$$

(iv) *The general even term is*

$$a_{2n} = \frac{3}{2n}\cdot\frac{3}{(2n-2)}\cdot\dots\cdot\frac{3}{2} = \frac{3^n}{(2n)(2n-2)\cdot\dots\cdot 2} = \frac{3^n}{2^n n!}.$$

Thus

$$y = \sum_{n=0}^{\infty}\frac{3^n}{2^n n!}x^{2n} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{3x^2}{2}\right)^n = \exp\left(\frac{3x^2}{2}\right).$$

Problem 6.

Consider the function

$$h(t) = \begin{cases} 2t + t^3 e^t & 0 \le t < 2 \\ t^2 + t^3 e^t & 2 \le t. \end{cases}$$

(i) Express h in terms of unit step functions.

(ii) Find the Laplace transform of h. You may leave your answer as a sum of fractions.

`Solution:`

(i) *We have*

$$h(t) = (2t + t^3 e^t) + (t^2 - 2t)u_2(t).$$

(ii) *The first term* $2t + t^3 e^t$ *has Laplace transform*

$$\frac{2}{s^2} + \frac{6}{(s-1)^4},$$

where the exponential shift formula was used above. For the second term, we write $(t^2 - 2t)u_2(t) = f(t-2)u_2(t)$ *where*

$$f(t-2) = t^2 - 2t \implies f(t) = (t+2)^2 - 2(t+2) = t^2 + 2t \implies F(s) = \frac{2}{s^3} + \frac{2}{s^2}.$$

Thus $(t^2 - 2t)u_2(t) = f(t-2)u_2(t)$ *has Laplace transform*

$$e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right).$$

Thus

$$H(s) = \left(\frac{2}{s^2} + \frac{6}{(s-1)^4}\right) + e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right).$$

Problem 7.

Use Laplace transforms to solve the initial value problem

$$y'' + 4y' + 5y = 10e^t, \;\; y(0) = 3, y'(0) = -2.$$

`Solution:` *Using Laplace transform, we find*

$$s^2Y - 3s + 2 + 4(sY - 3) + 5Y = \frac{10}{s-1}.$$

We solve

$$(s^2 + 4s + 5)Y = (3s + 10) + \frac{10}{s-1} = \frac{(3s+10)(s-1)+10}{s-1} = \frac{3s^2+7s}{s-1}.$$

Thus

$$Y(s) = \frac{3s^2+7s}{(s-1)(s^2+4s+5)}.$$

We write this into a sum of partial fractions

$$\frac{3s^2+7s}{(s-1)(s^2+4s+5)} = \frac{A}{s-1} + \frac{B(s+2)}{(s+2)^2+1} + \frac{C}{(s+2)^2+1}.$$

We solve for the undetermined coefficients

$$3s^2 + 7s = A(s^2+4s+5) + B(s+2)(s-1) + C(s-1) = (A+B)s^2 + (4A+B+C)s + (5A-2B-C)$$

$$\implies A + B = 3, 4A + B + C = 7, 5A - 2B - C = 0 \implies A = 1, B = 2, C = 1.$$

Thus

$$Y(s) = \frac{1}{s-1} + \frac{2(s+2)}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}$$

which yields

$$y(t) = e^t + 2e^{-2t}\cos t + e^{-2t}\sin t.$$

Problem 8.

Consider the forcing function

$$h(t) = u_1(t) + u_2(t).$$

(i) Solve the following initial value problem using Laplace transform

$$y'' - y = h(t),\ y(0) = y'(0) = 0.$$

(ii) Write your solution $y(t)$ explicitly over each of the three intervals

$$0 \le t < 1, \quad 1 \le t < 2, \quad 2 \le t < \infty.$$

`Solution:`

(i) *Using Laplace transform we obtain*

$$s^2 Y - Y = \frac{e^s}{s} + \frac{e^{2s}}{s} \implies Y(s) = \frac{e^{-s}}{s(s^2-1)} + \frac{e^{-2s}}{s(s^2-1)}.$$

We have

$$\frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}.$$

This gives

$$1 = A(s^2-1) + Bs(s+1) + Cs(s-1) = (A+B+C)s^2 + (B-C)s - A$$
$$\implies A+B+C = 0, B-C = 0, -A = 1 \implies A = -1, B = C = \frac{1}{2}.$$

Thus

$$\frac{1}{s(s^2-1)} = \frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1},$$

which comes via Laplace transform from the function

$$-1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}.$$

Thus

$$y(t) = u_1(t)\left(-1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1}\right) + u_2(t)\left(-1 + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-t+2}\right).$$

(ii) *For $t < 1$ we have $u_1(t) = u_2(t) = 0$ so*

$$y(t) = 0.$$

For $1 \le t < 2$ we have $u_1(t) = 1$ and $u_2(t) = 0$ so

$$y(t) = -1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1}.$$

For $t \ge 2$ we have $u_1(t) = u_2(t) = 1$ so

$$y = \left(-1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1}\right) + \left(-1 + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-t+2}\right) = -2 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1} + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-t+2}.$$