Problem 1.

The differential equation

$$(2x \sin y + y^2) + (x^2 \cos y + 2xy + e^y)y' = 0, \;\; y(0) = 0$$

is exact (you do not need to check this). Solve the differential equation and give the solution implicitly.

Write

$$M = 2x \sin y + y^2 \text{ and } N = x^2 \cos y + 2xy + e^y.$$

We search for a function Ψ such that

$$\Psi_x = 2x \sin y + y^2, \quad \Psi_y = x^2 \cos y + 2xy + e^y.$$

From the first equation we conclude

$$\Psi = x^2 \sin y + xy^2 + h(y).$$

Thus differentiating with respect to y we find

$$\Psi_y = 2x \cos y + 2xy + h'(y)$$

and comparing with $\Psi_y = x^2 \cos y + 2xy + e^y$ we obtain $h'(y) = e^y$. Hence $h(y) = e^y$ and

$$\Psi = x^2 \sin y + xy^2 + e^y.$$

The solution is constant along the potential function hence

$$x^2 \sin y + xy^2 + e^y = c.$$

Since $y(0) = 0$, by substitution we find $c = 1$. Hence the implicit solution is

$$x^2 \sin y + xy^2 + e^y = 1.$$

Problem 2.

Consider the autonomous differential equation

$$\frac{dy}{dt} = y^2 - 4y + 3.$$

Determine the critical points and indicate their type i.e. asymptotically stable, unstable, semistable. What is the long-term behavior of the solution satisfying the initial value $y(0) = 2$?

The critical points are found by solving

$$y^2 - 4y + 3 = 0 \implies (y-1)(y-3) = 0 \implies y = 1 \text{ or } y = 3.$$

The function $y^2 - 4y + 3$ has a graph a parabola which is seen to be positive for $y < 1$ and $y > 3$ and negative in the interval $1 < y < 3$. Looking at the derivative

$$\frac{dy}{dt} = y^2 - 4y + 3$$

we see that

- the derivative $\frac{dy}{dt}$ is positive for $y < 1$ and $y > 3$ so the function y is increasing for $y < 1$ and $y > 3$
- similarly, the solution y is decreasing for $1 < y < 3$.

Therefore, $y = 1$ is a stable critical point, while $y = 3$ is an unstable critical point.

The solution with $y(0) = 2$ will converge towards the stable critical value in the long run, so

$$\lim_{t\to\infty} y(t) = 1.$$

Problem 3.

Find the solution to the initial value problem

$$t^3y' + 5t^2y = 3, \;\; y(-1) = 2.$$

Where is the solution defined?

We write the equation in standard form

$$y' + \frac{5}{t}y = \frac{3}{t^3}.$$

We solve using an integrating factor

$$\mu = \exp \int \frac{5}{t}\, dt = \exp\left(5 \ln t\right) = t^5.$$

Multiplying both sides by the integrating factor yields

$$(t^5 y)' = t^5 \cdot \frac{3}{t^3} = 3t^2$$

which gives

$$t^5 y = t^3 + C.$$

Thus

$$y = \frac{1}{t^2} + \frac{C}{t^5}.$$

Since $y(-1) = 2$ we obtain $C = -1$ so that

$$y = \frac{1}{t^2} - \frac{1}{t^5}.$$

The solution is defined over the interval $(-\infty, 0)$.

Problem 4.

Write down the general solution of the differential equation

$$y'' + 4y' + 5y = 0.$$

Sketch the graph of the solution.

We form the characteristic equation

$$r^2 + 4r + 5 = 0$$

which has complex roots

$$r_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = -2 \pm i.$$

Thus the complex solutions are

$$y_1 = e^{(-2+i)t} = e^{-2t}(\cos t + i \sin t),\ \ y_2 = e^{(-2-i)t} = e^{-2t}(\cos t - i \sin t).$$

To find two real solutions we compute the real and the imaginary part of one of the two complex solutions. We obtain

$$u = e^{-2t} \cos t,\ \ v = e^{-2t} \sin t.$$

The general real value solution is

$$y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.$$

The graph is a damped oscillation whose amplitude decays as the exponential e^{-2t}.

Problem 5.

A swimming pool originally contains 200 gallons of water and 50 lb of salt. Water containing 4 lb of salt per gallon is poured into the pool at a rate of 2 gal/min. The mixture is allowed to leave the pool at the same rate. Find the amount of salt in the pool at time t.

Let $Q(t)$ be the amount of salt at time t. We write

$$\frac{dQ}{dt} = \text{Salt in} - \text{Salt out} = c_{\text{in}} \cdot \text{rate}_{\text{in}} - c_{\text{out}} \cdot \text{rate}_{\text{out}}.$$

Both rate in and rate out equal 2 gal/min. The concentration of salt going in is 4 lb/gal and $Q(t)/200$ lb/gal going out respectively. Substituting we find

$$\frac{dQ}{dt} = 2 \cdot 4 - 2 \cdot \frac{Q}{200} = 8 - \frac{Q}{100} = \frac{800 - Q}{100}.$$

This equation is separable. We obtain

$$\frac{dQ}{800 - Q} = \frac{dt}{100} \implies \int \frac{dQ}{800 - Q} = \int \frac{dt}{100} \implies -\ln(800 - Q) = \frac{t}{100} + C$$

which gives

$$800 - Q = Ke^{-t/100} \implies Q = 800 - Ke^{-t/100}.$$

The initial value is $Q(0) = 50$ which yields

$$800 - K = 50 \implies K = 750.$$

Thus

$$Q(t) = 800 - 750e^{-t/100}.$$