Problem 1.

Using undetermined coefficients, find a particular solution for the differential equation by undetermined coefficients

$$y'' - y' - 2y = 4e^{3t} + 5\sin t.$$

We seek a solution of the form

$$y = Ae^{3t} + B\sin t + C\cos t.$$

We compute

$$y' = 3Ae^{3t} - C\sin t + B\cos t$$
$$y'' = 9Ae^{3t} - B\sin t - C\cos t.$$

From here

$$y'' - y' - 2y = 4Ae^{3t} + (-3B + C)\sin t + (-3C - B)\cos t.$$

Since

$$y'' - y' - 2y = 4e^{3t} + 5\sin t$$

we can match coefficients to conclude

$$4A = 4 \implies A = 1$$
$$-3B + C = 5, -3C - B = 0 \implies B = -\frac{3}{2}, C = \frac{1}{2}.$$

Thus

$$y_p = e^{3t} - \frac{3}{2}\sin t + \frac{1}{2}\cos t.$$

Problem 2.

Find a particular solution for the following equation by variation of parameters

$$y'' - 6y' + 9y = \frac{e^{3t}}{t+1}.$$

We find the roots of the homogeneous equation $y'' - 6y' + 9y = 0$ first. To this end, we solve the characteristic equation $r^2 - 6r + 9 = 0$ which has a repeated root $r_1 = r_2 = 3$. The fundamental solutions are

$$y_1 = e^{3t}, y_2 = te^{3t}.$$

We compute the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (3t+1)e^{3t} \end{vmatrix} = e^{3t} \cdot (3t+1)e^{3t} - 3te^{3t} \cdot e^{3t} = e^{6t}.$$

Using undetermined coefficients we have

$$y = u_1 y_1 + u_2 y_2$$

where

$$u_1 = -\int \frac{te^{3t}}{e^{6t}} \cdot \frac{e^{3t}}{t+1}\, dt = -\int \frac{t}{t+1}\, dt = -\int \left(1 - \frac{1}{t+1}\right) dt = -\left(t - \ln(t+1)\right),$$

$$u_2 = \int \frac{e^{3t}}{e^{6t}} \cdot \frac{e^{3t}}{t+1}\, dt = \int \frac{dt}{t+1} = \ln(t+1).$$

Substituting, we find

$$y_p = -\left(t - \ln(t+1)\right) \cdot e^{3t} + \ln(t+1) \cdot te^{3t}.$$

Problem 3.

Consider the system

$$\vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}.$$

It is known that the matrix A has eigenvalues $\lambda = 2$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 4$ with eigenvector $\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(i) Write down a pair $\vec{x}_1, \vec{x}_2$ of fundamental solutions and verify that $W(\vec{x}_1, \vec{x}_2) \neq 0$.
(ii) Write down the general solution of the system.
(iii) Sketch a few of the trajectories and classify the type of critical point at the origin.

(i) We have

$$\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Thus

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} e^{2t} & 3e^{4t} \\ 0 & 2e^{4t} \end{vmatrix} = 2e^{6t} \neq 0.$$

(ii) The general solution is

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(iii) The origin is a source node.

- For $t \to -\infty$, the dominant term is $e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and solutions go to 0. Thus, solutions follow direction $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ when they are close to the origin.
- When $t \to \infty$, the dominant term is $e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and solutions follow the direction $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for large values.

Problem 4.

Consider the sytem

$$\vec{x}' = A\vec{x}, \;\; A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}.$$

(i) Write down the general solution.

(ii) Sketch a few of the trajectories. Clearly indicate the direction of the trajectory, and type of critical point at the origin.

(i) We find the eigenvalues and eigenvectors of the matrix. We have $\mathrm{Tr} A = 2, \det A = 5$ so the characteristic polynomial is

$$\lambda^2 - 2\lambda + 5 = 0 \implies \lambda = 1 \pm 2i.$$

We find the eigenvector for the eigenvalue $\lambda = 1 + 2i$. We have

$$A - (1+2i)I = \begin{bmatrix} -2-2i & 4 \\ -2 & 2-2i \end{bmatrix}$$

so one possible eigenvector is

$$v = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}.$$

We find one complex valued solution

$$\vec{x} = e^{t(1+2i)} \cdot \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

which then rewrites

$$\vec{x} = e^t(\cos 2t + i \sin 2t) \cdot \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = e^t \begin{bmatrix} 2\cos 2t + 2i \sin 2t \\ \cos 2t - \sin 2t + i(\cos 2t + \sin 2t) \end{bmatrix}.$$

The real and imaginary parts are

$$\vec{x}_1 = e^t \begin{bmatrix} 2\cos 2t \\ \cos 2t - \sin 2t \end{bmatrix}, \; \vec{x}_2 = e^t \begin{bmatrix} 2\sin 2t \\ \cos 2t + \sin 2t \end{bmatrix}.$$

The general solution is

$$\vec{x} = e^t \left(c_1 \begin{bmatrix} 2\cos 2t \\ \cos 2t - \sin 2t \end{bmatrix} + c_2 \begin{bmatrix} 2\cos 2t \\ \cos 2t - \sin 2t \end{bmatrix} \right).$$

There are other equivalent ways of expressing the answer.

(ii) The origin is a source spiral. To find the direction, we consider the initial value problem $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$\vec{x}'(0) = A\vec{x}(0) = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

This vector points downwards, so the spiral is clockwise.