Math 20D **Name:** ____________

August 23, 2018 **Midterm** **PID:** ____________

Instructions

1. **Write your *Name* and *PID* on the front of your Blue Book.**
2. **No calculators or other electronic devices are allowed during this exam.**
3. **You may use a double sided page of notes.**
4. **Write your solutions clearly in your Blue Book.**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order as they appear in the exam.**
   (c) **Start each numbered problem on a new side of a page.**
5. **Show all of your work and justify all your claims. No credit will be given for unsupported answers, even if correct.**

Complete 5 out of the 6 questions

1. (10 points) Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + x^2\cos(x)$$

Proof. This is a linear differential equation.

Rearrange into standard form

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right)y = x^2\cos(x)$$

Calculate integrating factor:

$$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln(x)} = \frac{1}{x}$$

Multiply by μ and simplify:

$$\frac{d}{dx}\left(\frac{1}{x}\cdot y\right) = x\cos(x)$$

Integrate

$$\begin{aligned}\frac{1}{x}\cdot y &= \int x\cos(x)dx + C\\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

General solution:

$$y = x(x\sin(x) + \cos(x) + C)$$

□

2. (10 points) Solve the initial value problem

$$\frac{dy}{dx} = (x+y)^2 - (x-y)^2, \quad y(1) = e^2$$

Proof. Once you simplify the right hand side it becomes clear that this is a separable equation:

$$\frac{dy}{dx} = 4xy$$

Separate variables and integrate:

$$\int \frac{1}{y} dy = \int 4x + C$$

Calculating we get:

$$\ln|y| = 2x^2 + C$$

We are given that $y > 0$ so we simplify $\ln|y| = \ln(y)$ and can write the solution as:

$$y = Ae^{2x^2}$$

Applying the initial value we get:

$$e^2 = Ae^2 \implies A = 1$$

So the (explicit) solution is given by:

$$y = e^{2x^2}$$

□

3. (10 points) **[This question has multiple parts]**

   (a) Find the general solution $y_h(t)$ to the homogeneous differential equation

   $$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$$

   (b) Give the general form of a **particular solution** $y_p(t)$ to

   $$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = te^t \qquad (1)$$

   (You do not need to solve for any unknown constants)

   (c) Using (a) and (b) give the general solution to the non-homogeneous equation (1)

Proof. (a) The auxiliary equation is given by

$$r^2 + 6r + 9 = 0$$

which has $r = -3$ as a repeated root. So the general form of the homogeneous solution is given by:

$$y_h(t) = C_1e^{-3t} + C_2te^{-3t}$$

(b) Since $r = +1$ is not a root to the auxiliary equation the general form of the particular solution is given by

$$y_p(t) = (At + B)e^t$$

(c) The general solution is a sum of the homogeneous and particular solution:

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-3t} + C_2 t e^{-3t} + (At + B)e^t$$

□

4. (10 points) Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{2x - y}{x + y - 4}$$

Hint: It may be a good idea to rewrite this as an equation involving a differential form.

Proof. Rewriting we get:

$$(2x - y)dx - (x + y - 4)dy = 0$$

This determines:

$$M = 2x - y, \quad N = -(x + y - 4)$$

Verify exactness:

$$M_y = -1$$
$$N_x = -1$$

Integrating M with respect to x we get:

$$F(x, y) = x^2 - xy + g(y) \tag{2}$$

Differentiate with respect to y and compare with N to get:

$$-x + g'(y) = -x - y + 4 \implies g'(y) = -y + 4$$

Integrate $g'(y)$ and substitute into (2):

$$F(x, y) = x^2 - xy - \frac{1}{2}y^2 + 4y \tag{3}$$

The solutions are given (implicitly) by:

$$x^2 - xy - \frac{1}{2}y^2 + 4y = C$$

For some constants C

□

5. (10 points) Solve the initial value problem

$$y''(t) + 4y(t) = 4\sin(2t); \quad y(0) = 1, \ y'(0) = 3 \tag{4}$$

Hint: The general form of the particular solution to (4) is given by

$$y_p(t) = At\cos(2t)$$

Proof. The auxiliary equation is given by

$$r^2 + 4 = 0$$

and so has two complex roots $+2i$ and $-2i$. So the general homogeneous solution is given by:

$$y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

Using the hint we calculate:

$$\begin{aligned} y_p' &= A\cos(2t) - 2At\sin(2t) \\ y_p'' &= -2A\sin(2t) - 2A(\sin(2t) + 2t\cos(2t)) \end{aligned}$$

Substituting into the differential equation we get:

$$-4A = 4 \implies A = -1$$

The general solution then has the form:

$$y(t) = C_1\cos(2t) + C_2\sin(2t) - t\cos(2t)$$

To apply the initial conditions we need to know $y'(t)$ so differentiate:

$$y'(t) = -2C_1\sin(2t) + 2C_2\cos(2t) - (\cos(2t) - 2t\sin(2t))$$

Apply initial conditions at $t = 0$ to get:

$$\begin{aligned} C_1 &= 1 \\ 2C_2 - 1 = 3 \implies C_2 &= 2 \end{aligned}$$

□

6. (10 points) Solve the initial value problem for the **Cauchy-Euler equation**

$$t^2y''(t) + 7ty'(t) + 5y(t) = 0; \quad y(1) = -1,\ y'(1) = 13$$

Proof. The auxiliary equation is given by

$$r^2 + 6r + 5 = 0$$

and has distinct roots $r = -1, -5$. The general form of the homogeneous solution is then given by:

$$y(t) = C_1t^{-1} + C_2t^{-5}$$

To apply the initial conditions we need to know $y'(t)$ so differentiate:

$$y'(t) = -C_1t^{-2} - 5C_2t^{-6}$$

Apply initial conditions at $t = 1$ to get:

$$C_1 + C_2 = -1 - C_1 - 5C_2 = 13$$

So we get:

$$\begin{aligned} C_1 &= 2 \\ C_2 &= -3 \end{aligned}$$

□