Math 20D **Name:** ____________________

September 8, 2018 **Practice Final** **PID:** ____________________

Instructions

1. **Write your *Name* and *PID* on the front of your Blue Book.**
2. **No calculators or other electronic devices are allowed during this exam.**
3. **You may use a double sided page of notes.**
4. **Write your solutions clearly in your Blue Book.**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order as they appear in the exam.**
   (c) **Start each numbered problem on a new side of a page.**
5. **Show all of your work and justify all your claims. No credit will be given for unsupported answers, even if correct.**

Complete 10 out of the 11 questions

1. (10 points) Find the general solution for the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -t \\ 4-3t \\ 1-2t \end{pmatrix} \tag{1}$$

Solution: The general solution $\mathbf{x}$ to any non-homogeneous system as above is always a sum of the homogeneous solution $\mathbf{x}_h$ and a particular solution $\mathbf{x}_p$

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p \tag{2}$$

The homogeneous solution $\mathbf{x}_h$ is calculated by solving the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x} \tag{3}$$

To find a particular solution $\mathbf{x}_p$ we can use the method of undetermined coefficients. Since the vector function

$$\begin{pmatrix} -t \\ 4-3t \\ 1-2t \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} t$$

in (1) is linear (no powers of t greater than 1), the general form of a particular solution is given by:

$$\mathbf{x}_p = \mathbf{a} + \mathbf{b}t$$

for some vectors $\mathbf{a}$, $\mathbf{b}$. By substituting this into (1) we can solve for $\mathbf{a}$ and $\mathbf{b}$.

Finding the homogeneous solution:

The general homogeneous solution $\mathbf{x}_h$ is the general solution to the system (3).

To calculate the eigenvalues of $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ we determine the characteristic equation:

$$\det \begin{pmatrix} 1-r & 1 & 0 \\ 0 & 3-r & 0 \\ 0 & 0 & 2-r \end{pmatrix} = 0$$

Calculating the determinant on the left hand side (by expanding along the first column) we get:

$$\det\begin{pmatrix}1-r & 1 & 0\\ 0 & 3-r & 0\\ 0 & 0 & 2-r\end{pmatrix} = (1-r)\cdot\det\begin{pmatrix}3-r & 0\\ 0 & 2-r\end{pmatrix} = (1-r)(3-r)(2-r) = 0$$

So we have 3 distinct eigenvalues

$$r_1 = 1$$
$$r_2 = 2$$
$$r_3 = 3$$

Calculating eigenvalue $\mathbf{u_1}$ (corresponding to $r_1 = 1$):

$\mathbf{u_1}$ is any non-zero solution to the matrix equation:

$$\begin{pmatrix}0 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

(notice 1 subtracted from the leading diagonal)

Equivalently this can be rewritten as a system of equations:

$$0a + b + 0c = 0 \implies b = 0$$
$$0a + 2b + 0c = 0$$
$$0a + 0b + c = 0 \implies c = 0$$

Using the above:

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}a\\ 0\\ 0\end{pmatrix} = a\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Let $a = 1$ to get

$$\mathbf{u_1} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Calculating eigenvalue $\mathbf{u_2}$ (corresponding to $r_2 = 2$):

$\mathbf{u_2}$ is any non-zero solution to:

$$\begin{pmatrix}-1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

(notice 2 subtracted from the leading diagonal)

Equivalently this can be rewritten as a system of equations:

$$-a + b + 0c = 0 \implies a = b$$
$$0a + b + 0c = 0 \implies b = 0$$
$$0a + 0b + 0c = 0$$

Using the above:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $c = 1$ to get

$$\mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Calculating eigenvalue $\mathbf{u_3}$ (corresponding to $r_3 = 3$):

$\mathbf{u_3}$ is any non-zero solution to (notice 3 subtracted from the leading diagonal):

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The method of row reduction can be used to solve this. **Reference**

Equivalently this can be rewritten as a system of equations:

$$\begin{aligned} -2a + b + 0c = 0 &\implies b = 2a \\ 0a + 0b + 0c = 0 & \\ 0a + 0b - c = 0 &\implies c = 0 \end{aligned}$$

Using the above:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 2a \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Let $a = 1$ to get

$$\mathbf{u_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

The homogeneous solution is given by:

$$\begin{aligned} \mathbf{x}_h(t) &= c_1 \mathbf{u_1} e^{r_1 t} + c_2 \mathbf{u_2} e^{r_2 t} + c_3 \mathbf{u_3} e^{r_3 t} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} e^{3t} \end{aligned}$$

Finding a particular solution:

For the particular solution we substitute the guess $\mathbf{x}_p = \mathbf{a} + \mathbf{b}t$ into (1). We get:

$$\begin{aligned} \frac{d}{dt}(\mathbf{a} + \mathbf{b}t) &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} (\mathbf{a} + \mathbf{b}t) + \begin{pmatrix} -t \\ 4 - 3t \\ 1 - 2t \end{pmatrix} \\ \mathbf{b} &= \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \right] + \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{b} + \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} \right] t \end{aligned}$$

On the right hand side of the second line I distributed the matrix products and separated the expression into two components: constants and multiples of t.

Comparing components on the two sides of the equation we get:

$$\begin{aligned} \mathbf{b} &= \left[\begin{pmatrix}1 & 1 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}\mathbf{a} + \begin{pmatrix}0\\4\\1\end{pmatrix}\right] \\ \mathbf{0} &= \left[\begin{pmatrix}1 & 1 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}\mathbf{b} + \begin{pmatrix}-1\\-3\\-2\end{pmatrix}\right] \end{aligned} \tag{4}$$

Rearranging the second equation of (4) we get:

$$\begin{pmatrix}1 & 1 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}\mathbf{b} = \begin{pmatrix}1\\3\\2\end{pmatrix}$$

If we let $\mathbf{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ the above matrix equation represents the simultaneous equations:

$$\begin{aligned} b_1 + b_2 + 0b_3 &= 1 \\ 0b_1 + 3b_2 + 0b_3 &= 3 \implies b_2 = 1 \\ 0b_1 + 0b_2 + 2b_3 &= 2 \implies b_3 = 1 \end{aligned}$$

Substituting $b_2 = 1$ into the first equation we get $b_1 = 0$. So:

$$\mathbf{b} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

Now we can substitute $\mathbf{b}$ into the first equation of (4):

$$\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}\mathbf{a} + \begin{pmatrix}0\\4\\1\end{pmatrix}$$

Rearrange:

$$\begin{pmatrix}0\\-3\\0\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}\mathbf{a}$$

If we let $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ the above can be rewritten as:

$$\begin{aligned} a_1 + a_2 + 0a_3 &= 0 \\ 0a_1 + 3a_2 + 0a_3 &= -3 \implies a_2 = -1 \\ 0a_1 + 0a_2 + 2a_3 &= 0 \implies a_3 = 0 \end{aligned}$$

Substituting $a_2 = -1$ into the first equation we get $a_1 = 1$. So:

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

This determines a particular solution:

$$\begin{aligned} \mathbf{x}_p &= \mathbf{a} + \mathbf{b}t \\ &= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} t \end{aligned}$$

Putting everything together, the general solution is given by:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_h + \mathbf{x}_p \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} e^{3t} + \left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} t \right) \end{aligned}$$

□

2. (10 points) Solve the initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution: The first part of the solution is the same as Question 1.

Characteristic polynomial:

$$\det \begin{pmatrix} -r & 1 \\ -2 & 3-r \end{pmatrix} = 0 \implies r^2 - 3r + 2 = 0$$

□

Eigenvalues:

$r_1 = 1$, $r_2 = 2$

Calculating $\mathbf{u}_1$ (associated to $r_1 = 1$):

Need to solve:

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

As a system of simultaneous equation this is:

$$\begin{aligned} -a + b &= 0 \\ -2a + 2b &= 0 \end{aligned}$$

Rearranging the first equation gives $b = a$ so we can rewrite $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ as:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Any choice of b (except $b = 0$) will give us an eigenvector, since there are no fractions to cancel let $b = 1$ to get:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Calculating $\mathbf{u}_2$ (associated to $r_2 = 2$):

Need to solve:

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

As a system of simultaneous equation this is:

$$\begin{aligned} -2a + b &= 0 \\ -2a + b &= 0 \end{aligned}$$

Rearranging the first equation gives $b = 2a$ so we can rewrite $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ as:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 2a \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Any choice of a (except $a = 0$) will give us an eigenvector, since there are no fractions to cancel let $a = 1$ to get:

$$\mathbf{u_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The general solution is then given by:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

Or equivalently as:

$$\mathbf{x}(t) = \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{5}$$

Substituting $t = 0$ into (5) and using the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ we get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Interpreting this as a system of simultaneous equations we get:

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 + 2c_2 &= -1 \end{aligned}$$

Subtracting one equation from the other we get $c_2 = -2$. Substituting $c_2 = -2$ back we get $c_1 = 3$.

So the solution is given by:

$$\mathbf{x}(t) = 3\begin{pmatrix}1\\1\end{pmatrix}e^t - 2\begin{pmatrix}1\\2\end{pmatrix}e^{2t} = \begin{pmatrix}3e^t - 2e^{2t}\\3e^t - 4e^{2t}\end{pmatrix}$$

3. (10 points) Consider the following differential equation

$$y''(t) - 3y'(t) + 2y(t) = f(t)$$

Using the method of undetermined coefficients, determine the general form of a particular solution $y_p(t)$ in the following cases (**do not calculate the unknown constants**):

(a) $f(t) = t^2 + 1$

(b) $f(t) = te^t + t$

(c) $f(t) = \sin(t) + \cos(2t)$

(d) $f(t) = \sin(t)e^{2t}$

Solution: It is a good idea to start by calculating the auxiliary roots of the given differential equation, so we know when to add an extra t (or potentially t^2) into the solution. Here the auxiliary equation is $r^2 - 3r + 2 = 0$ with roots $r_1 = 1$, $r_2 = 2$. This means if we $f(t)$ includes $p(t)e^t$ or $p(t)e^{2t}$ where $p(t)$ is any polynomial we will need to add an extra t.

(a) $f(t)$ is a quadratic polynomial so the guess for the particular solution is the general quadratic function:

$$y_p(t) = At^2 + Bt + C$$

Note: If 0 was one of the auxiliary roots, the correct guess would be $y_p(t) = t(At^2+Bt+C)$

(b) $f(t)$ is a sum of two functions that we handle separately. te^t is a product of a linear polynomial t and an exponential e^t. The corresponding guess would normally be given by $(At + B)e^t$, **however** because 1 is an auxiliary root e^t is already a homogeneous solution and so we need to include an extra t. This means $(At+B)te^t$ is the correct guess corresponding to te^t.

The correct guess corresponding to t is just $Ct + D$ as normal.

Combining the two guesses into one sum we get:

$$y_p(t) = (At + B)te^t + (Ct + D)$$

(c) $f(t)$ is a sum of two trigonometric functions with different arguments ($2t$ and $3t$) so we treat these as separate.

The correct guess associated to $\sin(t)$ is $A\cos(t)+B\sin(t)$ and the correct guess associated to $\cos(2t)$ is $C\cos(2t) + D\sin(2t)$. Combining the two we get:

$$y_p(t) = A\cos(t) + B\sin(t) + C\cos(2t) + D\sin(2t)$$

(d) $f(t)$ is a product of a trigonometric function $\sin(t)$ and an exponential e^{2t} and so the corresponding guess is the general product of the two:

$$y_p(t) = A\cos(t)e^{2t} + B\sin(t)e^{2t}$$

(Remember that you need to include both sin and cos)
Notice that despite the fact that 2 is an auxiliary root we do not include an extra power of t for e^{2t}. We would add an extra t into the guess here only if $2 \pm i$ were the auxiliary roots.

□

4. (10 points) Solve the equation

$$(y^3 + 4e^x y)dx + (4e^x + 3y^2 x)dy = 0$$

Solution: This looks like an exact equation but we first **need to check for exactness**:

Here we have:

$$M(x, y) = y^3 + 4e^x y$$
$$N(x, y) = 4e^x + 3y^2 x$$

Need to check if:

$$M_y = N_x$$

Taking partial derivatives:

$$M_y(x, y) = 3y^2 + 4e^x$$
$$N_x(x, y) = 4e^x + 3y^2$$

So this is an exact equation.

We can now look for a solution of the form

$$F(x, y) = C$$

Where $F_x = M$ and $F_y = N$.

Integrating M with respect to x (and treating y as a constant) we get:

$$\int y^3 + 4e^x y \; dx = y^3 x + 4e^x y$$

Integrating N with respect to y (and treating x as a constant) we get:

$$\int 4e^x + 3y^2 x \; dy = 4e^x y + y^3 x$$

$F(x, y)$ is a sum of the unique terms in the above calculations. $y^3 x$ and $4e^x y$ both show up twice and so:

$$F(x, y) = y^3 x + 4e^x y$$

With solutions given by

$$y^3 x + 4e^x y = C$$

for any constant C. □

5. (10 points) Solve the initial value problem:

$$\frac{dy}{dx} - \frac{y}{x} = xe^x, \quad y(1) = e - 1$$

Solution: Since this is a differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where

$$P(x) = -\frac{1}{x}$$
$$Q(x) = xe^x$$

this is a **linear equation**.

This means we have to find the integrating factor μ, recall this is calculated as:

$$\mu(x) = e^{\int P \ dx}$$

So in this case:

$$\begin{aligned}\mu(x) &= e^{\int -\frac{1}{x} \ dx} \\ &= e^{-\ln(x)} \\ &= e^{\ln(1/x)} \\ &= \frac{1}{x}\end{aligned}$$

Multiplying the differential equation by $\frac{1}{x}$ and simplifying gives:

$$\frac{d}{dx}\left(\frac{1}{x} \cdot y\right) = e^x \implies \frac{1}{x} \cdot y = \int e^x dx + C$$

Calculating the integral and solving for y we get:

$$y = x(e^x + C)$$

Applying the initial condition at $t = 1$:

$$(e - 1) = 1(e^1 + C) \implies C = -1$$

So the solution is:

$$y = x(e^x - 1)$$

□

6. (10 points) Find the general solution to the following equations:

(a)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$$

(b)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Solution: We need to determine and solve characteristic polynomial in each case:

(a) Characteristic Equation:

$$r^2 + 2r + 10 = 0 \implies (r+1)^2 + 9 = 0 \implies r = -1 \pm 3i$$

We need to use the form of the general solution corresponding to complex auxiliary roots $\alpha \pm i\beta$, recall that this is given by:

$$y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

In particular the general solution in our case corresponds to $\alpha = -1$, $\beta = 3$:

$$y(x) = c_1 e^{-x} \cos(3x) + c_2 e^{-x} \sin(3x)$$

(b) Characteristic Equation:

$$r^2 + 2r + 1 = 0 \implies (r+1)^2 = 0 \implies r = -1 \text{ (repeated)}$$

We need to use the form of the general solution corresponding to a repeated real root r, recall that this is given by:

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$

In particular the general solution in our case corresponds to $r = -1$:

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

□

7. (10 points) Find the general solution to the differential equation

$$y'' = 5x^{-1}y' - 13x^{-2}y, \quad x > 0$$

How would your answer change if we wanted a solution valid for $x < 0$?

Solution: This looks like a strange equation but is a Cauchy-Euler equation in disguise. If we multiply the equation through by x^2 and rearrange we get:

$$x^2y'' - 5xy' + 13y = 0, \quad x > 0$$

The characteristic equation corresponding to this Cauchy-Euler equation is given by:

$$r^2 + (-5-1)r + 13 = 0$$

Simplify and solve:

$$r^2 - 6r + 13 = 0 \implies (r-3)^2 + 4 = 0 \implies r = 3 \pm 2i$$

The general solution to a Cauchy-Euler equation with complex roots $\alpha \pm i\beta$ to the associated characteristic equation is given by:

$$y(x) = c_1 x^\alpha \cos(\beta \ln(x)) + c_2 x^\alpha \sin(\beta \ln(x))$$

In this question we have $\alpha = 3$, $\beta = 2$ and so the general solution (for $x > 0$) is given by:

$$y(x) = c_1 x^3 \cos(2 \ln(x)) + c_2 x^3 \sin(2 \ln(x))$$

If we wanted a solution valid for $x < 0$ we replace x with $-x$ to get:

$$y(x) = c_1(-x)^3 \cos(2\ln(-x)) + c_2(-x)^3 \sin(2\ln(-x))$$

□

8. (10 points) Using variation of parameters, find a particular solution to the differential equation

$$y'' - 2y' + y = \frac{e^t}{t}$$

Solution. We can only apply variation of parameters if we already have two linearly independent homogeneous solutions. Luckily here the left hand side has an easy auxiliary equation $r^2 - 2r + 1 = 0$ with $r = 1$ as a repeated root. This means we have two linearly independent solutions:

$$\begin{aligned} y_1(t) &= e^t \\ y_2(t) &= te^t \end{aligned}$$

This let's us set up the particular solution as:

$$\begin{aligned} y_p(t) &= v_1(t)y_1(t) + v_2(t)y_2(t) \\ &= v_1(t)e^t + v_2(t)te^t \end{aligned}$$

where $v_1(t)$ and $v_2(t)$ are determined by the equations:

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= \frac{f}{a} \end{aligned} \tag{6}$$

Here $f = \frac{e^t}{t}$ is the non-homogeneous part of the differential equation and $a = 1$ is the coefficient of y''.

We calculate $y_1' = e^t$, $y_2' = e^t + te^t$ and substitute into (8) to get:

$$\begin{aligned} e^t v_1' + te^t v_2' &= 0 \\ e^t v_1' + (e^t + te^t) v_2' &= \frac{e^t}{t} \end{aligned}$$

Subtract one equation from another to get:

$$e^t v_2' = \frac{e^t}{t} \implies v_2' = \frac{1}{t} \implies v_2 = \ln|t|$$

Substituting $v_2' = \frac{1}{t}$ into the first equation we get:

$$e^t v_1' + te^t \left(\frac{1}{t}\right) = 0 \implies v_1' = -1 \implies v_1 = -t$$

So a particular solution is given by:

$$\begin{aligned} y_p(t) &= v_1(t)e^t + v_2(t)te^t \\ &= -te^t + \ln|t|te^t \\ &= te^t(\ln|t| - 1) \end{aligned}$$

Note: the question does not specify whether $t > 0$ or $t < 0$, so we need to keep the absolute value sign in $\ln|t|$ throughout.

□

9. (10 points) Find a general solution to the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x(t) - 4y(t) \\ \frac{dy}{dt} &= x(t) + y(t) \end{aligned}$$

Solution: We can rewrite this as a system of differential equations $\mathbf{x}' = A\mathbf{x}$ with:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$$

The solution starts in the same way as in Question 1.

Characteristic polynomial:

$$\det \begin{pmatrix} 1-r & -4 \\ 1 & 1-r \end{pmatrix} = 0 \implies (1-r)^2 + 4 = 0$$

Eigenvalues:

$1 + 2i$, $1 - 2i \implies \alpha = 1, \beta = 2$

Calculating complex eigenvector:

The complex eigenvalue $\mathbf{z} = \mathbf{a} + i\mathbf{b}$ (where $\mathbf{a}$ and $\mathbf{b}$ are real vectors) associated to the eigenvalue $1 + 2i$ is calculated by solving:

$$(A - (1+2i)I)\mathbf{z} = \mathbf{0}$$

for $\mathbf{u}$. In this case we are solving:

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

As a system of equations this is:

$$\begin{aligned} -2ia - 4b &= 0 \\ a - 2ib &= 0 \end{aligned}$$

Rearranging the first equation:

$$b = \frac{-2ia}{4} = \left(-\frac{1}{2}i\right) a$$

The solution is then given by:

$$\mathbf{z} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ \left(-\frac{1}{2}i\right) a \end{pmatrix} = a \begin{pmatrix} 1 \\ -\frac{1}{2}i \end{pmatrix}$$

for any non-zero constant a.

Any choice of a (except 0) gives us an eigenvector, so to get rid of fractions let $a = 2$ and get:

$$\mathbf{z} = \begin{pmatrix} 2 \\ -i \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

So:

$$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

From this we can write down the general solution:

$$\begin{aligned}
\mathbf{x} &= c_1 e^{\alpha t} \left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) + c_2 e^{\alpha t} \left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right) \\
&= c_1 e^t \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin(2t) \right) + c_2 e^t \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(2t) \right) \\
&= c_1 e^t \left(\begin{pmatrix} 2\cos(2t) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -\sin(2t) \end{pmatrix} \right) + c_2 e^t \left(\begin{pmatrix} 2\sin(2t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\cos(2t) \end{pmatrix} \right) \\
&= c_1 e^t \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} 2\sin(2t) \\ -\cos(2t) \end{pmatrix}
\end{aligned}$$

□

10. (10 points) Solve the initial value problem

$$\frac{dy}{dx} - (1 + y^2)\tan(x) = 0, \quad y(0) = \sqrt{3}$$

Proof. Rearranging the equation we get:

$$\frac{dy}{dx} = (1 + y^2)\tan(x), \quad y(0) = \sqrt{3}$$

Notice that the right hand side function is conveniently separated into a product of the type $f(x) \cdot g(y)$ so we can identify this as a separated equation.

Rearranging the equation we get:

$$\frac{dy}{1 + y^2} = \tan(x)dx$$

Integrate both sides (recall you only need $+C$ on one side):

$$\int \frac{dy}{1+y^2} = \int \tan(x)dx + C \tag{7}$$

Both of these are standard integrals:

$$\int \frac{1}{1+y^2}\ dy = \arctan(y)$$
$$\int \arctan(x)\ dx = -\ln|\cos(x)|$$

Substituting these into (7) above we get:

$$\arctan(y) = -\ln|\cos(x)| + C \implies y = \tan(C - \ln|\cos(x)|)$$

We are given an initial condition at $x = 0$ so we are looking for a solution near $x = 0$ where $\cos(x) > 0$. This means that

$$\ln|\cos(x)| = \ln(\cos(x))$$

Applying the initial condition we can solve for C:

$$\sqrt{3} = \tan(C - \ln|\cos(0)|) = \tan(C) \implies C = \frac{\pi}{3}$$

So the solution is:

$$y = \tan\left(\frac{\pi}{3} - \ln|\cos(x)|\right)$$

□

11. (10 points) (a) Verify that $S = \left\{ \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}, \begin{pmatrix} e^{3t} \\ -2e^{3t} \end{pmatrix} \right\}$ is a fundamental solution set to the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \mathbf{x} \tag{8}$$

(b) Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Proof. (a) We need to check two things:

Part 1) Check that each function $\mathbf{x}(t)$ in S is a solution to the system (8)
Part 2) Check whether these are **linearly independent** solutions

For Part 1) we check whether the functions in S are solutions by substituting them into the system (8).

For example, let $\mathbf{x} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$ and substitute into (8):

$$\text{LHS} = \frac{d}{dt}\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix}$$
$$\text{RHS} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix}$$

Since LHS = RHS we know that $\mathbf{x} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$ is a solution. Similarly we can perform the same calculation with $\mathbf{x} = \begin{pmatrix} e^{3t} \\ -2e^{3t} \end{pmatrix}$ to check whether it is a solution. Instead of showing this calculation I want to highlight a method that can check both at the same time by verifying whether

$$X' = AX$$

holds, where X is the fundamental matrix:

$$X = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}$$

This calculation looks like this:

$$\text{LHS} = X' = \frac{d}{dt}\begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} & 3e^{3t} \\ -2e^{2t} & -6e^{3t} \end{pmatrix}$$
$$\text{RHS} = AX = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} & 3e^{3t} \\ -2e^{2t} & -6e^{3t} \end{pmatrix}$$

Since LHS = RHS both of the columns of X we solutions to $\mathbf{x}' = A\mathbf{x}$.

For Part 2) we can **use the Wronskian to check for linear independence**. Recall that the Wronskian is defined as the determinant of the fundamental matrix X.

$$\det(X) = \det\begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix} = -2e^{5t} + e^{5t} = -e^{5t}$$

Since $-e^{5t} \neq 0$ for all t the solutions from part (a) are linearly independent.

(b) We checked that the functions in S are two linearly independent solutions to the system (8), so any solution to (8) can be expressed as a linear combination of these two functions. We just need to determine the right coefficients to match the initial condition.

The general solution is given by:

$$\mathbf{x}(t) = c_1\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} + c_2\begin{pmatrix} e^{3t} \\ -2e^{3t} \end{pmatrix}$$

Apply the initial condition at $t = 0$:

$$\mathbf{x}(0) = c_1\begin{pmatrix} e^{0} \\ -e^{0} \end{pmatrix} + c_2\begin{pmatrix} e^{0} \\ -2e^{0} \end{pmatrix}$$

On the other hand we have:

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Combining the two:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

This can be expressed a system of simultaneous equations:

$$\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 - 2c_2 &= 1 \end{aligned}$$

Add the equations to get $-c_2 = 1 \implies c_2 = -1$
Substitute back in the first equation to get $c_1 = 1$
The solution is then given by:

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ -2e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} - \begin{pmatrix} e^{3t} \\ -2e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} - e^{3t} \\ -e^{2t} + 2e^{3t} \end{pmatrix} \end{aligned}$$

□