Math 20D - Spring 2017 - Final Exam

Name: ______________________________

Student ID: ______________________________

Section time: ______________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books, calculators or telephones. You may use a "cheat sheet" of notes which should be at most one page.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

You have 3 hours to complete the test.

Question	Score	Maximum
1		10
2		10
3		12
4		20
5		15
6		8
7		13
8		12
Total		100

Problem 1. [*10 points.*]

Using undetermined coefficients, find a general solution for the differential equation

$$y'' - 2y' = e^{2t} - 4t.$$

Problem 2. [*10 points.*]

Using integrating factors, find the general solution of the differential equation

$$ty' = t \cos t^4 - 3y.$$

Problem 3. [*12 points; 6, 6.*]

Consider the differential equation

$$x^2y'' - 2xy' + (2 - x^2)y = x^3e^x.$$

(i) Find the values of r for which $y = xe^{rx}$ is a solution to the *homogeneous* equation.

(ii) Using variation of parameters, find a particular solution to the *inhomogeneous* equation.

Hint: You may use as input $y_1 = xe^x$ and $y_2 = xe^{-x}$.

Problem 4. [*20 points; 4, 4, 4, 2, 6.*]

Consider the system $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}.$$

(i) Find a fundamental pair of solutions to the system.

(ii) Carefully draw a few trajectories (in particular, make sure you indicate their direction). Classify the type of critical point at the origin.

(iii) Calculate the normalized fundamental matrix $\Phi(t)$ with $\Phi(0) = I$.

(iv) Solve the initial value problem $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(v) Use variation of parameters to find a particular solution for the following inhomogeneous system

$$\vec{x}' = Ax + \begin{bmatrix} te^{3t} \\ 0 \end{bmatrix}.$$

Problem 5. [*15 points; 2, 6, 3, 4.*]

Consider the differential equation

$$y'' - 3xy' - 3y = 0 \text{ with initial conditions } y(0) = 1, y'(0) = 0$$

whose solution is written as a power series

$$y = a_0 + a_1 x + a_2 x^2 + \dots.$$

(i) Using the initial conditions, find a_0 and a_1.

(ii) Find the recurrence relation between the coefficients of the power series y.

(iii) Write down the first four *non-zero* terms of the solution. Is the solution even or odd?

(iv) Write down the general expression for the non-zero coefficients. Express the solution y in closed form. The final answer should be in terms of familiar functions.

Hint: You may need to recall the series expansion

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots + \frac{w^n}{n!} + \ldots.$$

Problem 6. [*8 points; 3, 5.*]

Consider the function

$$h(t) = \begin{cases} 2t + t^3 e^t & \text{if } 0 \leq t < 2 \\ t^2 + t^3 e^t & \text{if } 2 \leq t. \end{cases}$$

(i) Express h in terms of unit step functions.

(ii) Find the Laplace transform of h. You may leave your answer as a sum of fractions.

Problem 7. [*13 points.*]

Use Laplace transforms to solve the initial value problem

$$y'' + 4y' + 5y = 10e^t, \; y(0) = 3, y'(0) = -2.$$

Problem 8. [*12 points; 8, 4.*]

Consider the forcing function

$$h(t) = u_1(t) + u_2(t).$$

(i) Solve the following initial value problem using Laplace transform

$$y'' - y = h(t), \;\; y(0) = y'(0) = 0.$$

(ii) Write your solution $y(t)$ explicitly over each of the three intervals

$$0 \leq t < 1, \quad 1 \leq t < 2, \quad 2 \leq t < \infty.$$