Cauchy-Euler Equations

Recall that the general 2nd order linear differential equation is given by:

$$a(t)y'' + b(t)y' + c(t)y = f(t) \tag{1}$$

We have seen that when $a(t)$, $b(t)$ and $c(t)$ are constant functions (i.e. just constants) we can solve the homogeneous equation corresponding to (1):

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{2}$$

By the method summarized here.

For general functions $a(t)$, $b(t)$, $c(t)$ finding homogeneous solutions to (2) is very difficult. In fact, we have seen that using the method of variation of parameters (explained here) we can use the homogeneous solutions to (2) to construct particular solutions to (1). This means that in some sense the hard part is finding the homogeneous solutions to differential equations.

In this section we learn how to find homogeneous solutions in the next simplest kind of second order differential equation that is **equidimensional**, meaning that we have:

$$\begin{aligned} a(t) &= at^2 \\ b(t) &= bt \\ c(t) &= c \end{aligned}$$

where a, b, c are now constants. Differential equations of this type are also called **Cauchy-Euler** equations.

The method of solving them is very similar to the method of solving constant coefficient homogeneous equations. We set up a quadratic equation determined by the constants a, b, c, called the **characteristic equation**:

$$\alpha r^2 + (\beta - \alpha)r + \gamma = 0 \tag{3}$$

Homogeneous solutions to (2) are determined by the roots of (3). As before, there are 3 different cases depending on the type of roots (3) has:

1. 2 distinct real roots r_1 and r_2

2. 1 repeated real root r
3. 2 complex (conjugate) roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$

These correspond to following general homogeneous solutions:

For $t > 0$:

$$
\begin{aligned}
y_h(t) &= C_1 t^{r_1} + C_2 t^{r_2} && \text{(Case 1)}\\
y_h(t) &= C_1 t^{r} + C_2 \ln(t) t^{r} && \text{(Case 2)}\\
y_h(t) &= C_1 t^{\alpha} \cos(\beta \ln(t)) + C_2 t^{\alpha} \sin(\beta \ln(t)) && \text{(Case 3)}
\end{aligned}
$$

For $t < 0$:

$$
\begin{aligned}
y_h(t) &= C_1 (-t)^{r_1} + C_2 (-t)^{r_2} && \text{(Case 1)}\\
y_h(t) &= C_1 (-t)^{r} + C_2 \ln(-t)(-t)^{r} && \text{(Case 2)}\\
y_h(t) &= C_1 (-t)^{\alpha} \cos(\beta \ln(-t)) + C_2 (-t)^{\alpha} \sin(\beta \ln(-t)) && \text{(Case 3)}
\end{aligned}
$$

Solution Algorithm (Finding homogeneous solutions):

1. Determine the characteristic equation and solve it.
2. Use the roots of the auxiliary equation to decide which case you are in and use the corresponding form of the general homogeneous solution $y(t)$.

   Note: This depends on whether you are looking for a solution valid for $t > 0$ or $t < 0$. This will either be clear from the question statement (see the examples below), or you can deduce this from the initial conditions. If the initial conditions are given at some positive time t (i.e. $y(1) = 4$) then use the $t > 0$ solution, otherwise use the $t < 0$ solution.
3. (If given) apply initial conditions to solve for the constants C_1 and C_2.

Examples:

Example (4.7.13)**.** *Find the general solution to the given Cauchy-Euler equation for $t > 0$*

$$9t^2 y''(t) + 15ty'(t) + y(t) = 0 \tag{4}$$

Solution: The coefficients of (4) determine the characteristic equation:

$$9r^2 + 6r^2 + 1 = 0$$

Factorizing we get $(3r+1)^2 = 0$ which has a repeated solution $r = -\frac{1}{3}$.

So we use the general solution form corresponding to case 2 with $r = -\frac{1}{3}$ and $t > 0$. This means the general solution to (4) is given by:

$$y_h(t) = C_1 t^{-\frac{1}{3}} + C_2 \ln(t) t^{-\frac{1}{3}}$$

We have no initial conditions so we cannot determine C_1, C_2 $\square$

Example (4.7.19)**.** *Solve the given initial value problem for the Cauchy-Euler equation:*

$$t^2 y''(t) - 4t^2 y'(t) + 4y(t) = 0; \quad y(1) = -2, \quad y'(1) = -11 \tag{5}$$

Solution: As before we begin by solving the characteristic equation associated to (5):

$$r^2 - 5r + 4 = 0$$

This factorizes as $(r-1)(r-4) = 0$ and so has distinct solutions $r = 1$ and $r = 4$.

So we use the general solution form corresponding to case 1 with $r_1 = 1$, $r_2 = 2$ and since the initial condition was given for positive t (i.e. at $t = 1$) we want the solution to be valid for $t > 0$. This means the general solution to (5) is given by:

$$y_h(t) = C_1 t + C_2 t^4 \tag{6}$$

To make use of the second initial condition we need to differentiate once:

$$y_h'(t) = C_1 + 4C_2 t^3 \tag{7}$$

Applying the initial condition $y(1) = -2$ to (6) we get:

$$-2 = C_1 + C_2$$

Applying the initial condition $y'(1) = -11$ to (7) we get:

$$-11 = C_1 + 4C_2$$

Solving for C_1 and C_2 we get:

$$C_1 = 1, C_2 = -3$$

So the solution to the initial value problem (5) is given by:

$$y_h(t) = t - 3t^4$$

$\square$