2nd Order Constant Coefficient Differential Equations (Homogeneous Case)

The general form of a 2nd order differential equation is:

$$a(t)y'' + b(t)y' + c(t)y = f(t) \tag{1}$$

The simplest possible case (and the first we consider) is when a, b, c are constant functions and $f(t) = 0$ (the so called **homogeneous case**)

The solution to (1) is determined by the auxiliary equation

$$ar^2 + ar + b = 0 \tag{2}$$

There are 3 different cases determined by the roots of (2):

1. 2 distinct real roots r_1 and r_2
2. 1 repeated real root r
3. 2 complex (conjugate) roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$

These correspond to following general solutions:

$$\begin{aligned} y(t) &= C_1e^{r_1t} + C_2e^{r_2t} && \text{(Case 1)} \\ y(t) &= C_1e^{rt} + C_2te^{rt} && \text{(Case 2)} \\ y(t) &= C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha t}\sin(\beta t) && \text{(Case 3)} \end{aligned}$$

Solution Algorithm:

1. Determine the auxiliary equation and solve it.
2. Use the roots of the auxiliary equation to decide which case you are in and use the corresponding form of the general solution $y(t)$.
3. (If given) apply initial conditions to solve for the constants C_1 and C_2.

Example (4.2.3)**.** *Find a general solution to the given differential equation*

$$y'' + 5y' + 6y = 0 \tag{3}$$

Solution: The auxiliary equation is

$$r^2 + 5r + 6 = 0$$

and can be factorized as

$$(r + 3)(r + 2) = 0$$

so we have two distinct roots $r_1 = -2$ and $r_2 = -3$. Applying Case 1 we get the general solution:

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t}$$

(We have no given initial conditions so we cannot solve for C_1, C_2) □

Example (4.2.9)**.** *Find a general solution to the given differential equation*

$$4y'' - 4y' + y = 0 \tag{4}$$

Solution: The auxiliary equation is

$$4r^2 - 4r + 1 = 0$$

and can be factorized as

$$(2r - 1)^2 = 0$$

so we have the repeated root $r = \frac{1}{2}$. Applying Case 2 we get the general solution:

$$y(t) = C_1 e^{\frac{1}{2}t} + C_2 t e^{\frac{1}{2}t}$$

(We have no given initial conditions so we cannot solve for C_1, C_2) □

Example (4.3.13)**.** *Find a general solution to the given differential equation*

$$y'' - 2y' + 26y = 0 \tag{5}$$

Solution: The auxiliary equation is

$$r^2 - 2r + 26 = 0$$

Completing the square we get

$$(r - 1)^2 + 25 = 0$$

So we have complex conjugate roots $r_1 = 1 + 5i$ and $r_1 = 1 - 5i$. Applying Case 3 we get the general solution:

$$y(t) = C_1 e^t \cos(5t) + C_2 e^t \sin(5t)$$

□

Example (4.2.15). *Solve the initial value problem*

$$y'' - 4y' + 3y = 0 \quad y(0) = 1,\ y'(0) = \frac{1}{3} \tag{6}$$

Solution: The auxiliary equation is

$$r^2 - 4r + 3 = 0$$

and can be factorized as

$$(r-3)(r-1) = 0$$

so we have two distinct roots $r_1 = 1$ and $r_2 = 3$. Applying Case 1 we get the general solution:

$$y(t) = C_1 e^t + C_2 e^{3t} \tag{7}$$

To make use the initial conditions we need to differentiate $y(t)$ from (7) to get:

$$y'(t) = C_1 e^t + 3C_2 e^{3t}$$

Apply the initial conditions:

$$\begin{aligned} y(0) = 1 \quad &\Longrightarrow C_1 e^0 + C_2 e^0 = 1 &&\Longrightarrow C_1 + C_2 = 1 \\ y'(0) = \frac{1}{3} \quad &\Longrightarrow C_1 e^0 + 3C_2 e^0 = \frac{1}{3} &&\Longrightarrow C_1 + 3C_2 = \frac{1}{3} \end{aligned}$$

A simple calculation gives $C_1 = \frac{4}{3}$ and $C_2 = -\frac{1}{3}$.

So the (explicit) solution to (6) is given by:

$$y(t) = \frac{4}{3}e^t - \frac{1}{3}e^{3t} \tag{8}$$

□

Example (4.2.17). *Solve the initial value problem*

$$y'' - 6y' + 9y = 0 \quad y(0) = 2,\ y'(0) = \frac{25}{3} \tag{9}$$

Solution: The auxiliary equation is

$$r^2 - 6r + 9 = 0$$

and can be factorized as

$$(r-3)^2 = 0$$

so we have a repeated root $r = 3$. Applying Case 2 we get the general solution:

$$y(t) = C_1 e^{3t} + C_2 t e^{3t} \tag{10}$$

To make use the initial conditions we need to differentiate $y(t)$ from (10) to get:

$$y'(t) = 3C_1 e^{t} + C_2 \left(e^{3t} + 3te^{3t}\right)$$

Apply the initial conditions:

$$\begin{aligned} y(0) &= 2 && \Longrightarrow C_1 e^0 + 0 = 2 && \Longrightarrow C_1 = 2 \\ y'(0) &= \frac{25}{3} && \Longrightarrow 3C_1 e^0 + C_2\left(e^0 + 0\right) = \frac{25}{3} && \Longrightarrow 3C_1 + C_2 = \frac{25}{3} \end{aligned}$$

A simple calculation gives $C_2 = \frac{7}{3}$.

So the (explicit) solution to (6) is given by:

$$y(t) = 2e^{3t} + \frac{7}{3} t e^{3t} \tag{11}$$

□

Example (4.3.25)**.** *Solve the initial value problem*

$$y'' - 2y' + 2y = 0 \qquad y(\pi) = e^{\pi},\ y'(\pi) = 0 \tag{12}$$

Solution: The auxiliary equation is

$$r^2 - 2r + 2 = 0$$

Completing the square we get

$$(r-1)^2 + 1 = 0$$

So we have complex conjugate roots $r_1 = 1 + i$ and $r_1 = 1 - i$. Applying Case 3 we get the general solution

$$y(t) = C_1 e^t \cos(t) + C_2 e^t \sin(t) \tag{13}$$

Differentiate $y(t)$ from (13) to get:

$$y'(t) = C_1 \left(e^t \cos(t) - e^t \sin(t)\right) + C_2 \left(e^t \sin(t) + e^t \cos(t)\right)$$

Apply initial conditions:

$$\begin{aligned} y(\pi) = e^\pi \quad &\Longrightarrow C_1 e^\pi(-1) + 0 = e^\pi &&\Longrightarrow C_1 = -1 \\ y'(\pi) = 0 \quad &\Longrightarrow C_1(-e^\pi) + C_2(-e^\pi) = 0 &&\Longrightarrow C_2 = 1 \end{aligned}$$

So the (explicit) solution to (12) is given by:

$$y(t) = e^t \sin(t) - e^t \cos(t) = e^t(\sin(t) - \cos(t)) \tag{14}$$

□