Exact Differential Equations

It is difficult to define what exactly a differential form *is* so for us a **differential form** will simply mean a mathematical expression of the form:

$$M(x,y)dx + N(x,y)dy$$

A differential form is called **exact** if there is a function $F(x,y)$ such that:

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

It is not at all obvious which differential forms are exact. Luckily **we have a test for exactness:**

A differential form

$$M(x,y)dx + N(x,y)dy$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

A differential equation that can be rewritten as:

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

where $M(x,y)dx + N(x,y)dy$ is exact is called an **exact equation**.

Subtlety: There may be multiple ways to rewrite a particular differential equation into the form given by (1). Some of these may be exact while others may not! For example we can rewrite (1) as:

$$\frac{M(x,y)}{N(x,y)}dx + dy = 0$$

(This is almost never going to be exact)

Solution Algorithm:

1. Rewrite the differential equation in the form

$$M(x,y)dx + N(x,y)dy = 0$$

to identify M and N.

Note: Be careful with negative signs! If you have an exact equation:

$$2xdx - 2ydy = 0$$

then $N = -2y$.

2. Determine if $M(x,y)dx + N(x,y)dy$ is exact by applying the test for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

holds. If the differential form is exact we may proceed (if not we need to apply a different method).

3. There is a choice of either integrating $M(x,y)$ with respect to x or $N(x,y)$ with respect to y. Pick the easier of the two (for the rest of the solution I will assume we start by integrating M to get:

$$F(x,y) := \int M(x,y)dx + g(y) \tag{2}$$

When calculating the integral above treat y as a constant and notice that $g(y)$ replaces the usual constant of integration.

4. We need to determine $g(y)$. Take the partial derivative of $F(x,y)$ from (2) with respect to y to get

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + g'(y) \tag{3}$$

Because the differential equation is exact $N(x,y) = \frac{\partial F}{\partial y}$ so we can compare (3) with N to determine $g'(y)$.

5. Integrate $g'(y)$ to get $g(y)$ (there is no need to add $+C$ here)

6. Substitute the newly calculated $g(y)$ into (2) to determine F. The (implicit) solution to the differential equation (1) is given by

$$F(x,y) = C$$

for some constant C.

7. If given an initial condition, solve for C

Example (2.4.15). *Determine whether the equation is exact. If it is, then solve it.*

$$\cos(\theta)dr - (r\sin(\theta) - e^{\theta})d\theta = 0 \tag{4}$$

Solution: The variables are different in this question so it's important not to get confused. If we relabel $x = r$ and $y = \theta$ we can identify:

$$M(r,\theta) = \cos(\theta) \quad N(r,\theta) = e^{\theta} - r\sin(\theta)$$

Notice the sign change for $N(r,\theta)$!

We need to check for exactness by calculating:

$$\frac{\partial M}{\partial \theta} = -\sin(\theta)$$
$$\frac{\partial N}{\partial r} = -\sin(\theta)$$

We get equality so the differential form $\cos(\theta)dr - (r\sin(\theta) - e^{\theta})d\theta$ is exact. We may proceed with the solution.

At this state we have a choice of integrating M (with respect to r) or N (with respect to θ). In this question it seems easier to integrate M so we integrate:

$$\begin{aligned} F(r,\theta) &= \int \cos(\theta)dr + g(\theta) \\ &= r\cos(\theta) + g(\theta) \end{aligned}$$

Take the partial derivative with respect to θ:

$$\frac{\partial F}{\partial \theta} = -r\sin(\theta) + g'(\theta)$$

Compare with $N(r,\theta)$:

$$-r\sin(\theta) + g'(\theta) = e^{\theta} - r\sin(\theta) \implies g'(\theta) = e^{\theta}$$

Integrate

$$g(\theta) = \int e^{\theta} d\theta = e^{\theta}$$

So the solution to (4) is given (implicitly) by:

$$r\cos(\theta) + e^{\theta} = C$$

(We have no given initial condition so we cannot solve for C) □

Example (2.4.25)**.** *Solve the initial value problem*

$$(y^2 \sin(x))dx + (1/x - y/x)dy = 0, \quad y(\pi) = 1 \tag{5}$$

Solution: Identify M and N and test for exactness:

$$\begin{aligned} M(x,y) &= y^2 \sin(x) & \implies \frac{\partial M}{\partial y} &= 2y\sin(x) \\ N(x,y) &= 1/x - y/x & \implies \frac{\partial N}{\partial x} &= -\frac{1}{x^2} - \frac{y}{x^2} \end{aligned} \tag{6}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ this is not an exact differential equation so we cannot solve it with the above method. It is however separable:

$$(y^2 \sin(x))dx + (1/x - y/x)dy = 0 \tag{7}$$

can be rewritten as

$$(x\sin(x))dx = \left(\frac{y-1}{y^2}\right)dy \tag{8}$$

Integrate:

$$\int (x\sin(x))dx = \int \left(\frac{y-1}{y^2}\right)dy + C \tag{9}$$

The left hand side can be solved with integration by parts:

$$\begin{aligned} \int (x\sin(x))dx &= -x\cos(x) - \int (-\cos(x))\,dx \\ &= -x\cos(x) + \sin(x) \end{aligned} \tag{10}$$

The right hand should be split up as a sum of two integrals:

$$\begin{aligned} \int \left(\frac{y-1}{y^2}\right)dy &= \int \frac{1}{y} - \frac{1}{y^2}dy \\ &= \ln|y| + \frac{1}{y} \end{aligned} \tag{11}$$

Substitute back into (9):

$$-x\cos(x) + \sin(x) = \ln|y| + \frac{1}{y} + C \tag{12}$$

Use the initial condition ($y(\pi) = 1$) to solve for C:

$$-(\pi)(-1) + 0 = 0 + 1 + C \implies C = \pi - 1 \tag{13}$$

The (implicit) solution is then given by

$$-x\cos(x) + \sin(x) = \ln(y) + \frac{1}{y} + \pi - 1 \tag{14}$$

At this last step we got rid of the absolute value in the logarithm because we are looking for solutions near the initial condition where y is positive so $|y| = y$ (if the initial condition was given as $y(\pi) = -1$ then y is negative near the initial condition so would have replaced $|y|$ with $-y$). □

Example (2.4.21)**.** *Solve the initial value problem*

$$(1/x + 2y^2x)dx + (2yx^2 - \cos(y))dy = 0, \quad y(1) = \pi \tag{15}$$

Solution: Identify M and N and test for exactness:

$$M(x,y) = 1/x + 2y^2x \quad \Longrightarrow \frac{\partial M}{\partial y} = 4xy$$
$$N(x,y) = 2yx^2 - \cos(y) \quad \Longrightarrow \frac{\partial N}{\partial x} = 4xy$$

We get equality so the differential form $(1/x + 2y^2x)dx + (2yx^2 - \cos(y))dy$ is exact. We may proceed with the solution.

It seems easier to integrate N with respect to y to get $F(x,y)$ (note the $+g(x)$):

$$F(x,y) = \int 2yx^2 - \cos(y)dy + g(x) = x^2y^2 - \sin(y) + g(x)$$

Now differentiate with respect to x:

$$\frac{\partial F}{\partial x} = 2xy^2 + g'(x)$$

Comparing with M we have:

$$g'(x) = \frac{1}{x} \Longrightarrow g(x) = \ln|x|$$

Because we want a solution near the initial condition $y(1) = \pi$ (where $x = 1 > 0$) we have $|x| = x$ and so $\ln|x| = \ln(x)$.

The (implicit) solutions to (15) is given by the level curves

$$x^2y^2 - \sin(y) + \ln(x) = C$$

Applying the initial condition:

$$(1)^2(\pi)^2 - \sin(\pi) + \ln(1) = C \implies C = \pi^2$$

The (implicit) solution is then given by:

$$x^2y^2 - \sin(y) + \ln(x) = \pi^2 \tag{16}$$

□