# First Order Linear Equations

Linear first order differential equations are the second type of differential equations we've studied. At least in the first order case, linear differential equations can be solved with relatively ease by introducing a function called the integrating factor. The idea of the solution is that multiplying by the integrating factor helps us rewrite the differential equation in a particularly simple form using the product rule. Then we just have to integrate.

**Definition:** A (first order) differential equation that can be rewritten in the form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \tag{1}$$

is called **linear**

The important thing to notice here is that $a_1$, $a_0$ and $b$ are functions of **only** $x$, and that **there are no products** of $y$ and $\frac{dy}{dx}$ (with themselves or each other).

Here are some examples of linear and non-linear equations (obviously this list is by no means exhaustive):

| Linear | Non-linear |
|---|---|
| $\frac{dy}{dx} + y = 0$ | $\frac{dy}{dx}y = 0$ |
| $\frac{dy}{dt} = \frac{yt}{\sqrt{1-t}}$ | $\frac{dy}{dt} = \frac{yt}{\sqrt{y-t}}$ |
| $\cos(x)\frac{dy}{dx} - \sin(x)y = e^{2x}$ | $\cos(x)\frac{dy}{dx} - \sin(y)x = e^{2x}$ |
| $\frac{dr}{d\theta} = \theta^2$ | $\left(\frac{dr}{d\theta}\right)^2 = \theta$ |

**Solution algorithm:**

1. Rewrite your differential equation so it is in **standard form**:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{2}$$

2. Calculate the **integrating factor** $\mu$:

$$\mu(x) = e^{\int P(x)dx} \tag{3}$$

**Note:** We **do not** need to include $+C$ in the integral when evaluating $\mu(x)$

3. Multiply (2) by $\mu(x)$ calculated in (3) to get:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \tag{4}$$

4. By construction $\mu$ has the property that $\mu'(x) = \mu(x)P(x)$ so equation (4) always simplifies to:

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)Q(x)$$

5. Integrate (this time you **do** need $+C$ on the right hand side)

$$\mu(x)y = \int \mu(x)Q(x)dx + C$$

6. Divide by $\mu(x)$ to get the (explicit) solution to (2):

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

7. If given an initial condition, solve for $C$

**Example** (2.3.15)**.** *Obtain a general solution to the equation*

$$(x^2+1)\frac{dy}{dx} + xy - x = 0 \tag{5}$$

*Solution:* This is not in standard form so we need to add $x$ to both sides and divide by $x^2+1$ to get:

$$\frac{dy}{dx} + \left(\frac{x}{x^2+1}\right)y = \frac{x}{x^2+1} \tag{6}$$

In this case we have:

$$P(x) = Q(x) = \frac{x}{x^2+1}$$

To calculate $\mu(x)$ we first need to evaluate the integral

$$\int \frac{x}{x^2+1}dx$$

Recall the standard integral:

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x)) \tag{7}$$

Applying (7) with $f(x) = x^2 + 1$ we get:

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \ln(x^2 + 1)$$

(Remember we **do not** need $+C$ here). This means that:

$$\mu(x) = e^{\frac{1}{2} \ln(x^2+1)}$$

We can simplify this by using logarithm rules

$$\frac{1}{2} \ln(x^2 + 1) = \ln(\sqrt{x^2 + 1})$$

Because the exponential and logarithm are inverse functions we get:

$$\mu(x) = e^{\ln(\sqrt{x^2+1})} = \sqrt{x^2 + 1}$$

Multiplying (6) by $\mu(x)$ and simplifying (these are steps 3 and 4 in the algorithm) we get:

$$\frac{d}{dx}\left(y\sqrt{x^2 + 1}\right) = \frac{x}{\sqrt{x^2 + 1}}$$

Integrating (remember we **do** need $+C$ here) we get:

$$y\sqrt{x^2 + 1} = \int \frac{x}{\sqrt{x^2 + 1}} dx + C \tag{8}$$

Notice that

$$\frac{d}{dx}\left(\sqrt{x^2 + 1}\right) = \frac{x}{\sqrt{x^2 + 1}} \implies \int \frac{x}{\sqrt{x^2 + 1}} dx = \sqrt{x^2 + 1}$$

Applying this to (8) we get:

$$y\sqrt{x^2 + 1} = \sqrt{x^2 + 1} + C \tag{9}$$

So:

$$y = 1 + \frac{C}{\sqrt{x^2 + 1}} \tag{10}$$

(We have no given initial condition so we cannot solve for $C$) $\square$

**Example** (2.3.21). *Solve the initial value problem*

$$(\cos(x))\frac{dy}{dx} + y\sin(x) = 2x\cos^2(x), \quad y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32} \tag{11}$$

*Solution:* Divide by $\cos(x)$ to get the equation into standard form:

$$\frac{dy}{dx} + y\tan(x) = 2x\cos(x) \tag{12}$$

So we have:

$$P(x) = \tan(x) \quad Q(x) = 2x\cos(x)$$

To calculate $\mu(x)$ we first need to evaluate the integral:

$$\int \tan(x)dx$$

This is a standard integral (found on any good list of common integrals) and evaluates to:

$$\int \tan(x)dx = -\ln(|\cos(x)|) = \ln(|\sec(x)|) = \ln(\sec(x)) \tag{13}$$

**Subtlety:** We can get rid of the absolute values in (13) because we are interested in a solution **near the initial condition**, in particular close to $x = \pi/4$. For values of $x$ near $\pi/4$ $\cos(x)$ (and therefore $\sec(x)$) is positive so

$$|\sec(x)| = \sec(x)$$

If we had an initial condition $y(\pi) = 1$ for example, $\cos(x)$ would be negative near $x = \pi$ and so we would have $|\sec(x)| = -\sec(x)$.

Going back to the solution, (13) allows us to evaluate:

$$\mu(x) = e^{\ln(\sec(x))} = \sec(x)$$

Multiplying (12) by $\mu(x)$ and simplifying we get:

$$\frac{d}{dx}\left(y\sec(x)\right) = 2x$$

Integrate and divide by $\mu(x)$ to get:

$$y = \frac{x^2 + C}{\sec(x)} = \cos(x)(x^2 + C) \tag{14}$$

Apply the initial condition

$$y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$$

To get:

$$\frac{-15\sqrt{2}\pi^2}{32} = \frac{\sqrt{2}}{2}\left(\frac{\pi^2}{16} + C\right) \implies C = -\pi^2 \tag{15}$$

The final (explicit) solution is then given by:

$$y = \cos(x)(x^2 - \pi^2) \tag{16}$$

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