Non-homogeneous Systems

Example 1. *Find the particular solution to the non-homogeneous system*

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4t \\ -4t - 2 \end{pmatrix} \tag{1}$$

Solution: Since the non-homogeneous part of the equation

$$\begin{pmatrix} 4t \\ -4t - 2 \end{pmatrix}$$

is linear the particular solution should have the general form:

$$\mathbf{x}_p(t) = \mathbf{a} + \mathbf{b}t$$

Substituting into the system (5) we get:

$$\begin{aligned} \frac{d}{dt}(\mathbf{a} + \mathbf{b}t) &= \begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} (\mathbf{a} + \mathbf{b}t) + \begin{pmatrix} 4t \\ -4t - 2 \end{pmatrix} \\ \mathbf{b} &= \left[\begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{b} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} \right] t \end{aligned}$$

Equate components:

$$\begin{aligned} \mathbf{b} &= \left[\begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right] \\ \mathbf{0} &= \left[\begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{b} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} \right] \end{aligned}$$

Rearrange the second matrix equation to:

$$\begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{b} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$$

Solving we get:

$$\mathbf{b} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Substitute back into the first equation:

$$\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

and rearrange:

$$\begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Solving we get:

$$\mathbf{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So the particular solution is given by:

$$\begin{aligned} \mathbf{x}_p(t) &= \mathbf{a} + \mathbf{b}t \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} t = \begin{pmatrix} 0 \\ -2t \end{pmatrix} \end{aligned}$$

Example 2. *This is Question 9 on the practice final (found here)*

Find the general solution for the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -t \\ 4 - 3t \\ 1 - 2t \end{pmatrix} \tag{2}$$

Solution: The general solution $\mathbf{x}$ to any non-homogeneous system as above is always a sum of the homogeneous solution $\mathbf{x}_h$ and a particular solution $\mathbf{x}_p$

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p \tag{3}$$

The homogeneous solution $\mathbf{x}_h$ is calculated by solving the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x} \tag{4}$$

To find a particular solution $\mathbf{x}_p$ we can use the method of undetermined coefficients. Since the vector function

$$\begin{pmatrix} -t \\ 4-3t \\ 1-2t \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} t$$

in (2) is linear (no powers of t greater than 1), the general form of a particular solution is given by:

$$\mathbf{x}_p = \mathbf{a} + \mathbf{b}t$$

for some vectors $\mathbf{a}$, $\mathbf{b}$. By substituting this into (2) we can solve for $\mathbf{a}$ and $\mathbf{b}$.

Finding the homogeneous solution:

The general homogeneous solution $\mathbf{x}_h$ is the general solution to the system (4).

To calculate the eigenvalues of $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ we determine the characteristic equation:

$$\det \begin{pmatrix} 1-r & 1 & 0 \\ 0 & 3-r & 0 \\ 0 & 0 & 2-r \end{pmatrix} = 0$$

Calculating the determinant on the left hand side (by expanding along the first column) we get:

$$\det \begin{pmatrix} 1-r & 1 & 0 \\ 0 & 3-r & 0 \\ 0 & 0 & 2-r \end{pmatrix} = (1-r) \cdot \det \begin{pmatrix} 3-r & 0 \\ 0 & 2-r \end{pmatrix} = (1-r)(3-r)(2-r) = 0$$

So we have 3 distinct eigenvalues

$$r_1 = 1$$
$$r_2 = 2$$
$$r_3 = 3$$

Calculating eigenvalue $\mathbf{u_1}$ (corresponding to $r_1 = 1$):

$\mathbf{u_1}$ is any non-zero solution to the matrix equation:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(notice 1 subtracted from the leading diagonal)

Equivalently this can be rewritten as a system of equations:

$$\begin{aligned} 0a + b + 0c = 0 &\implies b = 0 \\ 0a + 2b + 0c = 0 & \\ 0a + 0b + c = 0 &\implies c = 0 \end{aligned}$$

Using the above:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Let $a = 1$ to get

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Calculating eigenvalue $\mathbf{u_2}$ (corresponding to $r_2 = 2$):

$\mathbf{u_2}$ is any non-zero solution to:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(notice 2 subtracted from the leading diagonal)

Equivalently this can be rewritten as a system of equations:

$$\begin{aligned} -a + b + 0c = 0 &\implies a = b \\ 0a + b + 0c = 0 &\implies b = 0 \\ 0a + 0b + 0c = 0 & \end{aligned}$$

Using the above:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $c = 1$ to get

$$\mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Calculating eigenvalue $\mathbf{u_3}$ (corresponding to $r_3 = 3$):

$\mathbf{u_3}$ is any non-zero solution to (notice 3 subtracted from the leading diagonal):

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The method of row reduction can be used to solve this. **Reference**

Equivalently this can be rewritten as a system of equations:

$$\begin{aligned} -2a + b + 0c = 0 &\implies b = 2a \\ 0a + 0b + 0c = 0 & \\ 0a + 0b - c = 0 &\implies c = 0 \end{aligned}$$

Using the above:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 2a \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Let $a = 1$ to get

$$\mathbf{u_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

The homogeneous solution is given by:

$$\begin{aligned} \mathbf{x}_h(t) &= c_1 \mathbf{u_1} e^{r_1 t} + c_2 \mathbf{u_2} e^{r_2 t} + c_3 \mathbf{u_3} e^{r_3 t} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} e^{3t} \end{aligned}$$

Finding a particular solution:

For the particular solution we substitute the guess $\mathbf{x}_p = \mathbf{a} + \mathbf{b}t$ into (2). We get:

$$\frac{d}{dt}(\mathbf{a} + \mathbf{b}t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}(\mathbf{a} + \mathbf{b}t) + \begin{pmatrix} -t \\ 4 - 3t \\ 1 - 2t \end{pmatrix}$$

$$\mathbf{b} = \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{a} + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}\right] + \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{b} + \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}\right] t$$

On the right hand side of the second line I distributed the matrix products and separated the expression into two components: constants and multiples of t.

Comparing components on the two sides of the equation we get:

$$\begin{aligned} \mathbf{b} &= \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{a} + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}\right] \\ \mathbf{0} &= \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{b} + \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}\right] \end{aligned} \tag{5}$$

Rearranging the second equation of (5) we get:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

If we let $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ the above matrix equation represents the simultaneous equations:

$$\begin{aligned} b_1 + b_2 + 0b_3 &= 1 \\ 0b_1 + 3b_2 + 0b_3 &= 3 \implies b_2 = 1 \\ 0b_1 + 0b_2 + 2b_3 &= 2 \implies b_3 = 1 \end{aligned}$$

Substituting $b_2 = 1$ into the first equation we get $b_1 = 0$. So:

$$\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Now we can substitute $\mathbf{b}$ into the first equation of (5):

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$$

Rearrange:

$$\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{a}$$

If we let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ the above can be rewritten as:

$$\begin{aligned} a_1 + a_2 + 0a_3 &= 0 \\ 0a_1 + 3a_2 + 0a_3 &= -3 \implies a_2 = -1 \\ 0a_1 + 0a_2 + 2a_3 &= 0 \implies a_3 = 0 \end{aligned}$$

Substituting $a_2 = -1$ into the first equation we get $a_1 = 1$. So:

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

This determines a particular solution:

$$\begin{aligned} \mathbf{x}_p &= \mathbf{a} + \mathbf{b}t \\ &= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} t \end{aligned}$$

Putting everything together, the general solution is given by:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_h + \mathbf{x}_p \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} e^{3t} + \left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} t \right) \end{aligned}$$

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