Separable Equations

The first non-trivial examples of differential equations we will learn to solve are called **separable**. The main idea in solving these equations is to separate variables then integrate respectively.

Definition:

A first-order differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

is called **separable** if the right hand side function $f(x, y)$ can be written as a product

$$f(x, y) = g(x)h(y)$$

where g and h are functions of x **only** and y **only** respectively.

Some examples of separable and non-separable equations (by no means an exhaustive list):

Separable	Non-separable
$\frac{dx}{dt} = x^2t^2$	$\frac{dx}{dt} = x^2 + t^2$
$\frac{dx}{dt} = \frac{yt}{\sqrt{1-t}}$	$\frac{dx}{dt} = \frac{yt}{\sqrt{y-t}}$
$\frac{dy}{dx} = e^{2x+3y}$	$\frac{dy}{dx} = \log(2x + 3y)$
$\frac{dr}{d\theta} = \frac{r(e^\theta+3)}{5r\theta}$	$\frac{dr}{d\theta} = \frac{3(e^\theta+r)}{5r\theta}$
$\frac{dy}{dx} = xy + y + x + 1$	$\frac{dy}{dx} = xy + y + x$

Solution algorithm:

1. Rewrite your differential equation into the following separated form (if necessary):
$$\frac{dy}{dx} = g(x) \cdot h(y) \tag{1}$$
2. Multiply (1) by dx and divide by $h(y)$ to get
$$\frac{dy}{h(y)} = g(x) \cdot dx \tag{2}$$

3. The (implicit) solution is given by integrating (2) (**don't forget** $+C$ on the right hand side)

$$\int \frac{dy}{h(y)} = \int g(x) \cdot dx + C$$

4. If given an initial condition, solve for C

Example (2.2.13)**.** *Solve the equation*

$$\frac{dy}{dx} = 3x^2(1+y^2)^{3/2} \tag{3}$$

Solution: First we verify that this is a separable equation. This is easily seen to be the case with $g(x) = 3x^2$ and $h(y) = (1+y^2)^{3/2}$

Rearranging (3) we get:

$$\frac{dy}{(1+y^2)^{3/2}} = 3x^2 dx$$

Integrate both sides:

$$\int \frac{dy}{(1+y^2)^{3/2}} = \int 3x^2 dx + C \tag{4}$$

To evaluate (4) we need to calculate the left hand side which is a tricky integral. Fortunately it can be made easier with a substitution. Applying the substitution $y = \tan(u)$ we transform the integral in y into an integral in u in the following way:

$$\int \frac{dy}{(1+y^2)^{3/2}} \longmapsto \int \frac{du \cdot \sec^2(u)}{\sec^3(u)}$$

Simplifying:

$$\int \frac{du \cdot \sec^2(u)}{\sec^3(u)} = \int \cos(u) du = \sin(u)$$

We need to rewrite this as a function of y. Since $y = \tan(u)$ we have $u = \arctan(y)$, and so:

$$\sin(u) = \sin(\arctan(y))$$

We can go much further in simplifying this complicated trigonometric function. Notice that:

$$y = \tan(u) \implies y^2 = \tan^2(u) = \sec^2(u) - 1$$

Rearranging we get:

$$\sec^2(u) = y^2 + 1$$

So:

$$\cos^2(u) = \frac{1}{y^2 + 1} \tag{5}$$

Sine and cosine are related by the formula:

$$\sin^2(u) + \cos^2(u) = 1 \tag{6}$$

Using (5) and (6) we get:

$$\begin{aligned}\sin(u) &= \sqrt{1 - \cos^2(u)} \\ &= \sqrt{\frac{y^2}{y^2 + 1}} \\ &= \sqrt{1 - \frac{1}{y^2 + 1}} \\ &= \frac{y}{\sqrt{y^2 + 1}}\end{aligned}$$

This lets us finally calculate the left hand side of (4) to get the (implicit) solution.

$$\frac{y}{\sqrt{y^2 + 1}} - x^3 = C \tag{7}$$

(We have no given initial condition so we cannot solve for C) □

Example (2.2.22)**.** *Solve the initial value problem*

$$x^2 dx + 2y dy = 0, \quad y(0) = 2 \tag{8}$$

Solution: This problem is not stated in the standard $\frac{dy}{dx} = f(x, y)$ form, but notice that it is still separated (in fact this is more of less the equation we expect at the end of step 2). We rearrange:

$$2y dy = -x^2 dx$$

and integrate:

$$\int 2y dy = \int -x^2 dx + C$$

Calculating we get:

$$y^2 = -\frac{1}{3}x^3 + C$$

Using the initial condition we can calculate C by substituting $x = 0$, $y = 2$

$$(2)^2 = -\frac{1}{3}(0)^3 + C \implies C = 4$$

The (implicit) solution to (8) is then given by:

$$y^2 = 4 - \frac{1}{3}x^3 \tag{9}$$

Going further we can take the square root:

$$y = \pm\sqrt{4 - \frac{1}{3}x^3}$$

The initial condition $y(0) = 2$ **forces** us to take the positive square root here to get the explicit solution:

$$y = \sqrt{4 - \frac{1}{3}x^3} \tag{10}$$

□