# Homogeneous systems of differential equations (with complex eigenvalues)

The basic method for solving systems of differential equations such as

$$\mathbf{x}' = A\mathbf{x} \tag{1}$$

is the same whether the matrix has real or complex eigenvalues. First calculate eigenvalues of the matrix, then find the corresponding eigenvectors.

There are however some differences. In the complex case the eigenvalues are always in a conjugate pair

$$\alpha + i\beta, \alpha - i\beta$$

and associated to these eigenvalues are the (complex) eigenvectors

$$\mathbf{a} + i\mathbf{b}, \mathbf{a} - i\mathbf{b}$$

that are also conjugate. In practice this means we only have to do the eigenvector calculation **once** - each complex eigenvalue pair determines 2 (linearly independent) solutions:

$$\begin{aligned}\mathbf{x}_1(t) &= e^{\alpha t}\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right)\\ \mathbf{x}_2(t) &= e^{\alpha t}\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)\end{aligned}$$

So the general solution is then given by:

$$\mathbf{x}(t) = c_1 e^{\alpha t}\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) + c_2 e^{\alpha t}\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right)$$

Notice that the general solution in the complex case depends only on the (**real** numbers) $\alpha$, $\beta$ and the (**real** vectors) $\mathbf{a}$, $\mathbf{b}$. This is nice because then the final solution consists only of real-valued functions.

**Worked examples:**

**Example 1.** *Find the general solution to the system:*

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{x}$$

*Solution:* **Calculate eigenvalues:**

Determine the characteristic equation and solve:

$$\det\begin{pmatrix} -r & 0 & 1 \\ 0 & -r & -1 \\ 0 & 1 & -r \end{pmatrix} = 0$$

Calculating the determinant (by expanding along the left column) we get:

$$\det\begin{pmatrix} -r & 0 & 1 \\ 0 & -r & -1 \\ 0 & 1 & -r \end{pmatrix} = (-r)\cdot\det\begin{pmatrix} -r & -1 \\ 1 & -r \end{pmatrix} = (-r)(r^2+1)$$

Solving $(-r)(r^2+1) = 0$ we get 3 solutions. One real root:

$$r_1 = 0$$

and the complex conjugate roots:

$$r_2 = i, \quad r_3 = -i$$

The complex conjugate roots are written as $\alpha \pm i\beta$ so in our case $\alpha = 0$ and $\beta = 1$

The eigenvector associated to $r_1 = 0$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (check this)

To calculate the eigenvector associated to $r_2 = i$ we need to solve:

$$\begin{pmatrix} -i & 0 & 1 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Rewriting as a system of equations:

$$\begin{aligned} (-i)a + 0b + c &= 0 \\ 0a - ib - c &= 0 \\ 0a + b - ic &= 0 \end{aligned}$$

Rearranging the first equation we get $ia = c \implies a = -ic$

Rearranging the second equation we get $ib = -c \implies b = ic$

So the solution is given by:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -ic \\ ic \\ c \end{pmatrix} = c \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix}$$

Let $c = 1$ to get the eigenvector:

$$\begin{pmatrix} -i \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

So we have:

$$\mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

The general solution is then given by:

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \left( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin(t) \right) + c_3 \left( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin(t) + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos(t) \right) \\ &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ -\sin(t) \\ \cos(t) \end{pmatrix} + c_3 \begin{pmatrix} -\cos(t) \\ \cos(t) \\ \sin(t) \end{pmatrix} \end{aligned}$$

$\square$

**Example 2.** *This is Question 9 on the practice final (found here)*

*Find a general solution to the system of differential equations*

$$\frac{dx}{dt} = x(t) - 4y(t)$$
$$\frac{dy}{dt} = x(t) + y(t)$$

*Solution:* We can rewrite this as a system of differential equations $\mathbf{x}' = A\mathbf{x}$ with:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$$

The solution starts in the same way as in Question **??**.

**Characteristic polynomial:**

$$\det \begin{pmatrix} 1-r & -4 \\ 1 & 1-r \end{pmatrix} = 0 \implies (1-r)^2 + 4 = 0$$

**Eigenvalues:**

$1 + 2i,\ 1 - 2i \implies \alpha = 1, \beta = 2$

**Calculating complex eigenvector:**

The complex eigenvalue $\mathbf{z} = \mathbf{a} + i\mathbf{b}$ (where $\mathbf{a}$ and $\mathbf{b}$ are real vectors) associated to the eigenvalue $1 + 2i$ is calculated by solving:

$$(A - (1+2i)I)\mathbf{z} = \mathbf{0}$$

for $\mathbf{u}$. In this case we are solving:

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

As a system of equations this is:

$$\begin{aligned} -2ia - 4b &= 0 \\ a - 2ib &= 0 \end{aligned}$$

Rearranging the first equation:

$$b = \frac{-2ia}{4} = \left(-\frac{1}{2}i\right) a$$

The solution is then given by:

$$\mathbf{z} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ \left(-\frac{1}{2}i\right) a \end{pmatrix} = a \begin{pmatrix} 1 \\ -\frac{1}{2}i \end{pmatrix}$$

for any non-zero constant $a$.

Any choice of $a$ (except 0) gives us an eigenvector, so to get rid of fractions let $a = 2$ and get:

$$\mathbf{z} = \begin{pmatrix} 2 \\ -i \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

So:

$$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$
$$\mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

From this we can write down the general solution:

$$\begin{aligned}
\mathbf{x} &= c_1 e^{\alpha t} \left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) + c_2 e^{\alpha t} \left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right) \\
&= c_1 e^t \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin(2t)\right) + c_2 e^t \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(2t)\right) \\
&= c_1 e^t \left(\begin{pmatrix} 2\cos(2t) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -\sin(2t) \end{pmatrix}\right) + c_2 e^t \left(\begin{pmatrix} 2\sin(2t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\cos(2t) \end{pmatrix}\right) \\
&= c_1 e^t \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} 2\sin(2t) \\ -\cos(2t) \end{pmatrix}
\end{aligned}$$

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