Homogeneous systems of differential equations (with real eigenvalues)

The solutions to a homogeneous system of **differential** equations

$$\mathbf{x}' = A\mathbf{x} \tag{1}$$

(where A is a matrix with real eigenvalues) always have the form:

$$\mathbf{x} = \mathbf{u}e^{rt}$$

where r is an eigenvalue of A and $\mathbf{u}$ is its corresponding eigenvector. So **each eigenvalue eigenvector pair determines a solution**. If A is a 2×2 matrix we expect to get 2 solutions (corresponding to each eigenvalue) and if A is 3×3 we except to get 3.

The **general** solution then is given by:

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \tag{2}$$

(when A is a 2×2 matrix) or by

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 \tag{3}$$

(when A is a 3×3 matrix)

To solve system (1) we use the following algorithm:

Solution Algorithm:

1. Calculate the eigenvalues of A by solving the characteristic equation:

   $$\det(A - rI) = 0$$

   Note: This step requires you to calculate the **determinant** of a matrix.

2. Calculate the eigenvectors of A by solving the homogeneous matrix equation

   $$(A - rI)\mathbf{u} = \mathbf{0}$$

   for $\mathbf{u}$ by row reduction.

   Note: Alternatively you can rewrite $(A - rI)\mathbf{u} = \mathbf{0}$ as a system of simultaneous equations and solve by hand (there are examples of this in the handout).

3. Write down the general solution as in (2) or (3) (depending whether A is a 2×2 or 3×3 matrix).

4. (If given) use the initial condition $\mathbf{x}(t_0) = \mathbf{b}$ to solve for c_1, c_2, and c_3 (if A is 3×3).

   Note: This step requires you to solve the **non-homogeneous** matrix equation $A\mathbf{c} = \mathbf{b}$ for $\mathbf{c}$, or alternatively you can rewrite $A\mathbf{c} = \mathbf{b}$ as a system of simultaneous equations and solve that by hand. See Examples 2 and 4.

Worked Examples:

Example 1. *Find a general solution of the system*

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} \tag{4}$$

Solution: **Step 1: Find eigenvalues**

First we need to find the eigenvalues of $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, so set up the characteristic equation:

$$\det \begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} = 0$$

Calculating the determinant we get $(2-r)(-2-r)+3=0$ which simplifies to $r^2-1=0$. Solving we get the eigenvalues $r_1 = 1$ and $r_2 = -1$.

Step 2: Find eigenvectors

Calculating eigenvector $\mathbf{u_1}$ associated to $r_1 = 1$:

$\mathbf{u_1}$ is calculated as any (non-zero) solution to the matrix equation:

$$(A - r_1 \cdot I)\mathbf{u} = \mathbf{0}$$

Since $r_1 = 1$ we have:

$$A - 1 \cdot I = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$$

And so we want to solve the matrix equation:

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is equivalent to the simultaneous equations:

$$\begin{aligned} a - b &= 0 \\ 3a - 3b &= 0 \end{aligned}$$

From the first equation we get $b = a$, so we can rewrite $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ as:

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Any choice of a (except $a = 0$) will give us an eigenvector, since there are no fractions to cancel let $a = 1$ to get:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Calculating eigenvector $\mathbf{u_2}$ associated to $r_2 = -1$:

Similarly we find $\mathbf{u_2}$ by solving the matrix equation:

$$(A - r_2 \cdot I)\mathbf{u} = \mathbf{0}$$

Since $r_2 = -1$ we have:

$$A - (-1) \cdot I = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$$

So again we want to solve a matrix equation:

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is equivalent to the simultaneous equations:

$$\begin{aligned} 3a - b &= 0 \\ 3a - b &= 0 \end{aligned}$$

Rearranging the first equation we get $b = 3a$, so we can rewrite $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ as:

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 3a \end{pmatrix} = a \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Any choice of a (except $a = 0$) will give us an eigenvector, since there are no fractions to cancel let $a = 1$ to get:

$$\mathbf{u_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The general solution is then given by:

$$\mathbf{x}(t) = c_1 \mathbf{u_1} e^{r_1 t} + c_2 \mathbf{u_2} e^{r_2 t}$$

Substituting the constants we calculated above we get:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

This can also be rewritten as:

$$\mathbf{x}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Recall: $\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$ is the **fundamental matrix**.

We do not have any initial conditions so we cannot determine $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $\square$

Example 2 (9.5.31)**.** *Solve the initial value problem*

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{5}$$

Solution: **Step 1: Find eigenvalues**

First we find the eigenvalues of $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

Set up the characteristic equation:

$$\det \begin{pmatrix} 1-r & 3 \\ 3 & 1-r \end{pmatrix} = 0$$

Calculating the determinant we get $r^2 - 2r - 8 = 0$, which simplifies to $r^2 - 2r - 8 = 0$. Solving for r we get the eigenvalues $r_1 = 4$ and $r_2 = -2$.

Step 2: Find eigenvectors

Calculating eigenvector $\mathbf{u_1}$ associated to $r_1 = 4$:

$\mathbf{u_1}$ is any (non-zero) solution to $(A - 4 \cdot I)\mathbf{u} = \mathbf{0}$,

$$A - 4 \cdot I = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$$

So we want to solve

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is equivalent to the simultaneous equations:

$$\begin{aligned} -3a + 3b &= 0 \\ 3a - 3b &= 0 \end{aligned}$$

Rearranging the first equation we get $a = b$, so we can rewrite $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ as:

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Any choice of b (except $b = 0$) will give us an eigenvector, since there are no fractions to cancel let $b = 1$ to get:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Calculating eigenvector $\mathbf{u_2}$ associated to $r_2 = -2$:

$\mathbf{u_2}$ is the solution to $(A - (-2) \cdot I)\mathbf{u} = \mathbf{0}$,

$$A - (-2) \cdot I = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

So we want to solve

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is equivalent to the simultaneous equations:

$$\begin{aligned} 3a + 3b &= 0 \\ 3a + 3b &= 0 \end{aligned}$$

Rearranging the first equation we get $a = -b$, so we can rewrite $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ as:

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ b \end{pmatrix} = b \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Any choice of b (except $b = 0$) will give us an eigenvector, since there are no fractions to cancel let $b = 1$ to get:

$$\mathbf{u_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The general solution is given by:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

This can also be rewritten as:

$$\mathbf{x}(t) = \begin{pmatrix} e^{4t} & e^{-2t} \\ e^{4t} & -e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{6}$$

We can use the initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ to calculate $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ by substituting $t = 0$ into (6) we get:

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

This is equivalent to the simultaneous equations:

$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 - c_2 &= 1 \end{aligned} \tag{7}$$

We can use row reduction to solve this matrix equation or just solve it by hand.

Solving by row reduction:

Set up an augmented matrix and reduce:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right) \quad (R_2 \longmapsto R_2 - R_1)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right) \quad (R_2 \longmapsto -\frac{1}{2} \cdot R_2)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right) \quad (R_1 \longmapsto R_1 - R_2)$$

Now we can just read off the solution from the last column:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Solving by hand:

We can subtract the second equation of (7) from the first to get:

$$2c_2 = 2 \implies c_2 = 1$$

Now substitute $c_2 = 1$ back into the first equation to get:

$$c_1 = 2$$

So the solution to (5) is:

$$\begin{aligned} \mathbf{x}(t) &= 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} \\ &= \begin{pmatrix} 2e^{4t} + e^{-2t} \\ 2e^{4t} - e^{-2t} \end{pmatrix} \end{aligned}$$

□

Example 3 (9.5.15)**.** *Find a general solution of the system*

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix} \mathbf{x} \tag{8}$$

Solution: **Step 1: Finding eigenvalues**

The characteristic equation is given by:

$$\det\begin{pmatrix} 1-r & 2 & 3 \\ 0 & 1-r & 0 \\ 2 & 1 & 2-r \end{pmatrix} = 0$$

Calculating the determinant along the second row we get:

$$\begin{aligned} \det\begin{pmatrix} 1-r & 2 & 3 \\ 0 & 1-r & 0 \\ 2 & 1 & 2-r \end{pmatrix} &= (1-r)\cdot\det\begin{pmatrix} 1-r & 3 \\ 2 & 2-r \end{pmatrix} \\ &= (1-r)\left[(1-r)(2-r)-6\right] \\ &= (1-r)(r^2-3r-4) \\ &= (1-r)(r+1)(r-4) \end{aligned}$$

So we get 3 eigenvalues as solutions $r_1 = 1$, $r_2 = -1$, $r_3 = 4$ to

$$(1-r)(r-4)(r+1) = 0$$

Step 2: Finding eigenvectors

$\mathbf{u_1}$ (associated to $r_1 = 1$) is calculated by solving:

$$\begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is equivalent to the simultaneous equations:

$$\begin{aligned} 0a + 2b + 3c &= 0 \\ 0a + 0b + 0c &= 0 \\ 2a + b + c &= 0 \end{aligned} \tag{9}$$

Solving by row reduction:

Determine the augmented matrix:

$$(A|\mathbf{0}) = \left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

Apply row operations:

$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right) \quad (R_1 \longleftrightarrow R_3)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_2 \longleftrightarrow R_3)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto \frac{1}{2} \cdot R_1) \textbf{ and } (R_2 \longmapsto \frac{1}{2} \cdot R_2)$$

$$\left(\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto R_1 - \frac{1}{2} \cdot R_2)$$

Rewrite as a system of equations and solve:

$$\begin{aligned} a - \frac{1}{4}c = 0 \quad &\Longrightarrow a = \frac{1}{4}c \\ b + \frac{3}{2}c = 0 \quad &\Longrightarrow b = -\frac{3}{2}c \end{aligned}$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1/4 \cdot c \\ -3/2 \cdot c \\ 1 \cdot c \end{pmatrix} = c \begin{pmatrix} 1/4 \\ -3/2 \\ 1 \end{pmatrix}$$

Solving by hand:

Rearranging the first equation of (9) we get:

$$2b = -3c \implies b = -\frac{3}{2}c$$

Substitute into the last equation to get:

$$2a + \left(-\frac{3}{2}c\right) + c = 0 \implies 2a - \frac{1}{2}c = 0 \implies a = \frac{1}{4}c$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1/4 \cdot c \\ -3/2 \cdot c \\ 1 \cdot c \end{pmatrix} = c \begin{pmatrix} 1/4 \\ -3/2 \\ 1 \end{pmatrix}$$

Any non-zero choice of c will give us an eigenvector, to cancel fractions let $c = 4$ to get:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}$$

$\mathbf{u_2}$ (associated to $r_2 = -1$) is calculated by solving:

$$\begin{pmatrix} 2 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is equivalent to the simultaneous equations:

$$\begin{aligned} 2a + 2b + 3c &= 0 \\ 0a + 2b + 0c &= 0 \\ 2a + b + 3c &= 0 \end{aligned} \tag{10}$$

Solving by row reduction:

Determine the augmented matrix:

$$(A|\mathbf{0}) = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right)$$

Apply row operations:

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \quad (R_3 \longmapsto R_3 - R_1)$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto \frac{1}{2} \cdot R_1) \textbf{ and } (R_2 \longmapsto \frac{1}{2} \cdot R_2)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto R_1 - R_2) \textbf{ and } (R_3 \longmapsto R_3 + R_2)$$

Rewrite as a system of equations and solve:

$$\begin{aligned} a + \frac{3}{2}c = 0 \quad &\implies a = -\frac{3}{2}c \\ b = 0 \quad &\implies b = 0c \end{aligned}$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3/2 \cdot c \\ 0 \cdot c \\ 1 \cdot c \end{pmatrix} = c \begin{pmatrix} -3/2 \\ 0 \\ 1 \end{pmatrix}$$

Solving by hand:

The second equation of (10) immediately gives $b = 0$.

Substituting into the first equation we get:

$$2a + 2(0) + 3c = 0 \implies a = -\frac{3}{2}c$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}c \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$$

Any non-zero choice of c will give us an eigenvector, to cancel fractions let $c = 2$ to get:

$$\mathbf{u_2} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$

$\mathbf{u_3}$ (associated to $r_3 = 4$) is calculated by solving:

$$\begin{pmatrix} -3 & 2 & 3 \\ 0 & -3 & 0 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is equivalent to the simultaneous equations:

$$\begin{aligned} -3a + 2b + 3c &= 0 \\ 0a - 3b + 0c &= 0 \\ 2a + b - 2c &= 0 \end{aligned} \tag{11}$$

Solving by row reduction:

Determine the augmented matrix:

$$(A|\mathbf{0}) = \left(\begin{array}{ccc|c} -3 & 2 & 3 & 0 \\ 0 & -3 & 0 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

Apply row operations:

$$\left(\begin{array}{ccc|c} -3 & 2 & 3 & 0 \\ 0 & -3 & 0 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \quad (R_1 \longmapsto R_1 + 2 \cdot R_3) \textbf{ and } (R_2 \longmapsto -\frac{1}{3} \cdot R_2)$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -7 & 0 & 0 \end{array}\right) \quad (R_3 \longmapsto R_3 - 2 \cdot R_3)$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -7 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto R_1 - 4 \cdot R_2) \textbf{ and } (R_3 \longmapsto R_3 + 7 \cdot R_2)$$

Rewrite as a system of equations and solve:

$$\begin{aligned} a - c = 0 \quad &\implies a = c \\ b = 0 \quad &\implies b = 0 \cdot c \end{aligned}$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \cdot c \\ 0 \cdot c \\ 1 \cdot c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Solving by hand:

The middle equation of (11) gives $b = 0$

Substituting into the first equation we get:

$$-3a + (0) + 3c = 0 \implies a = c$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

There are no fractions to cancel so let $c = 1$ to get:

$$\mathbf{u_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The general solution is given by:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{4t}$$

This can also be rewritten as:

$$\mathbf{x}(t) = \begin{pmatrix} e^{t} & -3e^{-t} & e^{4t} \\ -6e^{t} & 0 & 0 \\ 4e^{t} & 2e^{-t} & e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

□

Example 4 (9.5.33)**.** *Solve the initial value problem*

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} \tag{12}$$

Solution: **Step 1: Finding eigenvalues**

The characteristic equation is given by:

$$\det \begin{pmatrix} 1-r & -2 & -2 \\ -2 & 1-r & -2 \\ 2 & -2 & 1-r \end{pmatrix} = 0$$

Calculating the determinant (along the first row) we get:

$$\begin{aligned}
&\det \begin{pmatrix} 1-r & -2 & -2 \\ -2 & 1-r & -2 \\ 2 & -2 & 1-r \end{pmatrix} \\
&= (1-r) \cdot \det \begin{pmatrix} 1-r & -2 \\ -2 & 1-r \end{pmatrix} - (-2) \cdot \det \begin{pmatrix} -2 & -2 \\ 2 & 1-r \end{pmatrix} + (-2) \cdot \det \begin{pmatrix} -2 & 1-r \\ 2 & -2 \end{pmatrix} \\
&= (1-r)(r^2 - 2r - 3) + 2(2r+2) - 2(2r+2) \\
&= (1-r)(r^2 - 2r - 3) \\
&= (1-r)(r+1)(r-3)
\end{aligned}$$

So we get 3 eigenvalues as solutions $r_1 = 1$, $r_2 = -1$, $r_3 = 3$ to

$$(1-r)(r+1)(r-3) = 0$$

Step 2: Finding eigenvectors

$\mathbf{u_1}$ (associated to $r_1 = 1$) is calculated by solving:

$$\begin{pmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving by row reduction:

Write down the augmented matrix:

$$(A|\mathbf{0}) = \left(\begin{array}{ccc|c} 0 & -2 & -2 & 0 \\ -2 & 0 & -2 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right)$$

Apply row operations:

$$\left(\begin{array}{ccc|c} 0 & -2 & -2 & 0 \\ -2 & 0 & -2 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \quad (R_1 \longleftrightarrow R_3)$$

$$\left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \quad (R_2 \longmapsto R_2 + R_1)$$

$$\left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_3 \longmapsto R_3 - R_2)$$

$$\left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto \frac{1}{2} \cdot R_1) \textbf{ and } (R_2 \longmapsto -\frac{1}{2} \cdot R_2))$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto R_1 + R_2)$$

Rewrite as a system of equations and solve:

$$\begin{aligned} a + c = 0 \quad &\Longrightarrow a = -c \\ b + c = 0 \quad &\Longrightarrow b = -c \end{aligned}$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -c \\ -c \\ c \end{pmatrix} = c \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Any non-zero choice of c will give us an eigenvector, there are no fractions to cancel so let $c = 1$ to get:

$$\mathbf{u_1} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$\mathbf{u_2}$ (associated to $r_2 = -1$) is calculated by solving:

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving by row reduction:

Write down the augmented matrix:

$$(A|\mathbf{0}) = \left(\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right)$$

Apply row operations:

$$\left(\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \quad (R_2 \longmapsto R_2 + R_1) \textbf{ and } (R_3 \longmapsto R_3 - R_1)$$

$$\left(\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_3 \longmapsto R_3 + R_2)$$

$$\left(\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto \frac{1}{2} \cdot R_1) \textbf{ and } (R_2 \longmapsto -\frac{1}{4} \cdot R_2)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto R_1 + R_2)$$

Rewrite as a system of equations and solve:

$$\begin{aligned} a - b &= 0 \quad \implies a = b \\ c &= 0 \end{aligned}$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ b \\ 0 \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Any non-zero choice of b will give us an eigenvector, there is no need to cancel fractions so let $b = 1$ to get:

$$\mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\mathbf{u_3}$ (associated to $r_3 = 3$) is calculated by solving:

$$\begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving by row reduction:

Determine the augmented matrix:

$$(A|\mathbf{0}) = \left(\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ 2 & -2 & -2 & 0 \end{array}\right)$$

Apply row operations:

$$\left(\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ 2 & -2 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right) \quad (R_2 \longmapsto R_2 - R_1) \textbf{ and } (R_3 \longmapsto R_3 + R_1)$$

$$\left(\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad (R_1 \longmapsto -\frac{1}{2} \cdot R_1) \textbf{ and } (R_3 \longmapsto -\frac{1}{4} \cdot R_3)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_2 \longleftrightarrow R_3)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_1 \longmapsto R_1 - R_2)$$

Rewrite as a system of equations and solve:

$$\begin{aligned} a &= 0 \\ b + c &= 0 \implies b = -c \end{aligned}$$

Write down the solution:

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -c \\ c \end{pmatrix} = c \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Any non-zero choice of c will give us an eigenvector, there is no need to cancel fractions so let $c = 1$ to get:

$$\mathbf{u_3} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The general solution is given by:

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{u_1} e^{r_1 t} + c_2 \mathbf{u_2} e^{r_2 t} + c_3 \mathbf{u_3} e^{r_3 t} \\ &= c_1 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{3t} \end{aligned}$$

This can also be rewritten as:

$$\mathbf{x}(t) = \begin{pmatrix} -e^{t} & e^{-t} & 0 \\ -e^{t} & e^{-t} & -e^{3t} \\ e^{t} & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{13}$$

We can now use the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$$

to solve for c_1, c_2, c_3. Substituting $t = 0$ into (13) we get the matrix equation:

$$\mathbf{x}(0) = \begin{pmatrix} -e^{0} & e^{0} & 0 \\ -e^{0} & e^{0} & -e^{0} \\ e^{0} & 0 & e^{0} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Since $\mathbf{x}(0) = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$ this simplifies to:

$$\begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Solving by row reduction:

Set up an augmented matrix and reduce:

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & -2 \\ -1 & 1 & -1 & -3 \\ 1 & 0 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -1 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad (R_2 \longmapsto R_2 - R_1) \text{ and } (R_3 \longmapsto R_3 + R_1)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right) \quad (R_2 \longleftrightarrow R_3) \text{ and } (R_1 \longmapsto -1 \cdot R_1)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad (R_1 \longmapsto R_1 + R_2) \text{ and } (R_3 \longmapsto -1 \cdot R_3)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad (R_1 \longmapsto R_1 - R_3) \text{ and } (R_2 \longmapsto R_2 - R_3)$$

Now we can just read off the solution from the last column:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

So the solution to (12) is:

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{3t} \\ &= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} e^t - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{3t} \\ &= \begin{pmatrix} -e^t - e^{-t} \\ -e^t - e^{-t} - e^{3t} \\ e^t + e^{3t} \end{pmatrix} \end{aligned}$$

$\square$