Terminology from Section 9

Abstract Preliminaries

A $m \times n$ **matrix** is a 2-dimensional array with m rows and n columns and will typically be denoted by upper case letters A, B, C etc.

Examples:

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} t & 2 & -t \\ 0 & t^2 & -t \\ 3 & -t & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Here A and C are **constant** matrices while B is not. A and B both have the same number of rows and columns, such matrices are called **square**.

Note: We will mostly be working with constant square matrices so most of the examples we discuss will be of this nature.

A **vector** is a special kind of matrix with either one row (called a **row vector**) or one column (called a **column vector**).

Examples:

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ -\frac{5}{2} \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} 9 & 2t & 0 & \sin(t) \end{pmatrix} \tag{1}$$

Here $\mathbf{v_1}$ is a 3×1 column vector, whereas $\mathbf{v_2}$ is a 1×4 row vector. Vectors can be constant (i.e. $\mathbf{v_1}$) or can have functions as entries (i.e. $\mathbf{v_2}$).

We can do calculations with matrices just as with real numbers. Adding and subtracting matrices is easy while matrix multiplication is little more complicated. If you have never worked with matrices or would like a refresher the following videos might be useful:

Khan Academy Video: Adding & subtracting matrices

Khan Academy Video: Intro to matrix multiplication

For each n there is a special square matrix called the $(n \times n)$ **identity** matrix, typically denoted by I_n or $I_{n\times n}$. You can think of these matrices as matrix versions of 1 because they have the property that:

$$AI_n = A = I_nA$$

for any $n\times n$ matrix A (compare this with the property 1 has: $x\cdot 1 = x = 1\cdot x$ for any number x).

Khan Academy Video: Intro to identity matrix

Examples:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The **determinant** of a matrix is a technical tool we use to determine some of the properties of the matrix. The general definition of the determinant is complicated and we will only calculate the determinant for 2×2 and 3×3 matrices so I will discuss these two cases:

Determinant of a 2×2 matrix:

Let A be the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A denoted det(A) or $|A|$, is given by:

$$\det(A) = ad - bc \tag{2}$$

Determinant of a 3×3 matrix:

Calculating the determinant of a 3×3 matrix is more complicated and involves calculating several 2×2 determinants. Let B be the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then det(B) is given by:

$$\begin{aligned} \det(B) &= a \cdot \det\left(\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right) - b \cdot \det\left(\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right) + c \cdot \det\left(\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right) \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei - afh - bdi + bfg + cdh - cge \end{aligned} \tag{3}$$

Note: Notice the change of sign in the coefficients in (3), the pattern is $+a$, $-b$, $+c$. This is because there are signs associated to each entry in a matrix. These are:

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix} \text{ for a } 2 \times 2 \text{ matrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \text{ for a } 3 \times 3 \text{ matrix}$$

The determinant formula in (3) was calculated by "expanding along **the top row**" (this is why the coefficients that show up are a, b, c).

There are equivalent formulae you can write down by expanding along any row or column. This can especially be useful if there is a row or column with mostly zeros.

Khan Academy Video: Determinant of a 3x3 matrix: standard method

Khan Academy Video: Determinant of a 3x3 matrix: shortcut method

Determinants of matrices have two useful properties:

1. **It tells us when a matrix are invertible (i.e has an inverse):**

$$A \text{ is invertible if and only if } \det(A) \neq 0$$

2. **It tells us when a collection of vectors is linearly independent:**
   Let $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ be a collection of (column) vectors and form a matrix A by concatenating the vectors into a single matrix[1]

$$\text{The columns of A are linearly independent if and only if } \det(A) \neq 0$$

Eigenvalues and Eigenvectors:

Let A be a matrix. A vector $\mathbf{u} \neq \mathbf{0}$ is called an **eigenvector** (of A) if there is a constant r such that:

$$A\mathbf{u} = r\mathbf{u}$$

[1]For example: If $\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} -1 \\ 4 \\ -7 \end{pmatrix}, A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -3 & 4 \\ -2 & -1 & -7 \end{pmatrix}$

then the constant r is called an **eigenvalue** (of A).

To calculate eigenvalues we solve the equation

$$\det(A - rI) = 0$$

If A is a 2×2 matrix this is a quadratic equation, if A is a 3×3 this is a cubic equation.

Given an eigenvalue r we find the associated eigenvectors by solving the matrix equation

$$(A - rI)\mathbf{u} = \mathbf{0}$$

Khan Academy Video: Introduction to eigenvalues and eigenvectors

Systems of Differential Equations

A **system** of differential equations consists of a collection of functions

$$x_1(t), x_2(t), \cdots, x_n(t)$$

with derivates that are related in the following **linear** system:

$$\begin{aligned}
x_1'(t) &= \frac{dx_1}{dt} = a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\
x_2'(t) &= \frac{dx_2}{dt} = a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\
&\vdots \\
x_n'(t) &= \frac{dx_n}{dt} = a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t)
\end{aligned} \tag{4}$$

Example:

$$\begin{aligned}
\frac{dx}{dt} &= x + y \\
\frac{dy}{dt} &= x - y
\end{aligned}$$

Using matrix algebra we can rewrite any linear system as (4) in a much more compact way. First we collect together the unknown functions $x_1(t), x_2(t), \cdots, x_n(t)$ into a column vector called the **solution vector**:

$$\mathbf{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

We denote the column vector of derivates by $\mathbf{x}'$:

$$\mathbf{x}'(t) := \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$$

The **coefficient matrix** records all the coefficient functions (most coefficient matrices we will see will be constant):

$$A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

We can also collect the non-homogeneous parts (the functions $f_i(t)$) of each equation in (4) as a column vector:

$$\mathbf{f}(t) := \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

With A, $\mathbf{x}$ and $\mathbf{f}$ as above, (4) can be rewritten as a **matrix equation**:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \tag{5}$$

A system of differential equations is in **normal form** when it is written this way.

When $\mathbf{f} \equiv \mathbf{0}$ (identically the zero vector) then we say that the system is **homogeneous**, otherwise it is **non-homogeneous**.

Note: It is important to be clear in the understanding that (4) and (5) are two different ways of writing the exact same system of equations.

The advantage of using matrix equations such as (5) is that:

1. It is a more compact representation of the problem.
2. It allows us to use tools from abstract linear algebra to look for solutions.

Example:

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = x - y$$

can be rewritten in normal form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ with:

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A set of vector-valued functions $S = \{\mathbf{x_1}(t), \mathbf{x_2}(t), \cdots, \mathbf{x_n}(t)\}$ is called a **fundamental solution set** to the system (5) if:

1. Each $\mathbf{x_i}(t)$ in S is a solution to (5) (i.e. if we substitute $\mathbf{x}(t) = \mathbf{x_i}(t)$ into the equation we get equality).
2. The vectors in S are linearly independent (i.e. the determinant of the matrix whose columns are the functions $\mathbf{x_1}(t), \mathbf{x_2}(t), \cdots, \mathbf{x_n}(t)$ is never 0)