2nd Order Constant Coefficient Differential Equations (Non-Homogeneous Case)

Method of Undetermined Coefficients

The first method we learnt to tackle non-homogeneous constant coefficient differential equations is called the method of undetermined coefficients. This is a fancy name for what amounts to trial and error used to determine the particular solution \( y_p \).

Consider the following differential equation:

\[
 ay'' + by' + cy = f(t) \tag{1}
\]

The function \( f(t) \) determines a guess for what the particular solution looks like (with some unknown coefficients). By substituting the guess into (1) we can solve for the coefficients and determine the \( y_p(t) \) exactly.

It is important to note that this method only works for specific functions \( f(t) \) that are made up of polynomials, exponentials and sinusoids (sines and cosines). For ease of notation let \( P_m(t) \) and \( Q_n(t) \) be polynomials of degree \( m \) and \( n \) respectively.

**Case 1:** \( f(t) = P_m(t)e^{\lambda t} \)

**Use the form:**

\[
 y_p(t) = t^s (A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0) e^{\lambda t} \tag{2}
\]

Where

\[
 s = \begin{cases} 
 0 & \text{if } \lambda \text{ is not a root of the auxiliary equation} \\
 1 & \text{if } \lambda \text{ is a root (not repeated) of the auxiliary equation} \\
 2 & \text{if } \lambda \text{ is a repeated root of the auxiliary equation} 
\end{cases}
\]

**Case 2:** \( f(t) = P_m(t)e^{\alpha t} \cos(\beta t) + Q_n(t)e^{\alpha t} \sin(\beta t) \)

**Use the form:**
\[ y_p(t) = t^s \left( A_k t^k + A_{k-1} t^{k-1} + \cdots + A_0 \right) e^{\alpha t} \cos(\beta t) \]
\[ + t^s \left( B_k t^k + B_{k-1} t^{k-1} + \cdots + B_0 \right) e^{\alpha t} \sin(\beta t) \quad (3) \]

Where \( k \) is the maximum of \( m \) and \( n \) (the degrees of the polynomials \( P_m(t) \) and \( Q_n(t) \) respectively) and

\[ s = \begin{cases} 
0 & \text{if } \alpha + i\beta \text{ is not a root of the auxiliary equation} \\
1 & \text{if } \alpha + i\beta \text{ is a root of the auxiliary equation} 
\end{cases} \]

\textbf{Note:} It is impossible for \( \alpha + i\beta \) to be a repeated root (since \( \alpha - i\beta \) is always the other root) so there is no option for \( s = 2 \).

\textbf{Note:} If you have a function \( f(t) \) such as \( \tan(t) \), \( \ln(2t) \), \( 1/t \) etc, that is not given by any of the above you need to use variation of parameters instead to solve your problem.
**Particular Solution General Form** (Distinct Roots):

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Aux. roots</th>
<th>$y_p(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y'' - 3y' + 2y = 4t^2 - 2$</td>
<td>1,2</td>
<td>$At^2 + Bt + C$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = \sin(t)$</td>
<td>1,2</td>
<td>$A \cos(t) + B \sin(t)$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = \sin(2t)$</td>
<td>1,2</td>
<td>$A \cos(2t) + B \sin(2t)$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = 12e^{3t}$</td>
<td>1,2</td>
<td>$Ae^{3t}$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = 2e^t$</td>
<td>1,2</td>
<td>$Ae^t$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = 4t^2 e^t$</td>
<td>1,2</td>
<td>$(At^2 + Bt + C)te^t$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = \frac{3}{4}t e^{3t}$</td>
<td>1,2</td>
<td>$(At + B)e^{3t}$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = t^3 - \cos(3t)$</td>
<td>1,2</td>
<td>$(At^3 + Bt^2 + Ct + D) + E \cos(3t) + F \sin(3t)$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = t^2 \cos(2t) + t^3 \sin(2t)$</td>
<td>1,2</td>
<td>$(At^3 + Bt^2 + Ct + D) \cos(2t) + (Et^3 + Ft^2 + Gt + H) \sin(2t)$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = e^{3t} \sin(t)$</td>
<td>1,2</td>
<td>$Ae^{3t} \cos(t) + Be^{3t} \sin(t)$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = e^{3t} + \sin(t)$</td>
<td>1,2</td>
<td>$Ae^{3t} + B \cos(t) + C \sin(t)$</td>
</tr>
<tr>
<td>$y'' - 3y' + 2y = e^t \sin(t)$</td>
<td>1,2</td>
<td>$Ae^t \cos(t) + Be^t \sin(t)$</td>
</tr>
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<td>$y'' - 3y' + 2y = e^t + \sin(t)$</td>
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### Particular Solution General Form (Repeated Root):

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<tr>
<td>$y'' + 2y' + y = 3t^2 + t$</td>
<td>-1, -1</td>
<td>$At^2 + Bt + C$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = \sin(t)$</td>
<td>-1, -1</td>
<td>$A\cos(t) + B\sin(t)$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = \sin(2t)$</td>
<td>-1, -1</td>
<td>$A\cos(2t) + B\sin(2t)$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = \pi^2 e^{3t}$</td>
<td>-1, -1</td>
<td>$Ae^{3t}$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = 5e^{-t}$</td>
<td>-1, -1</td>
<td>$At^2 e^{-t}$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = 7te^{-t}$</td>
<td>-1, -1</td>
<td>$(At + B)t^2 e^{-t}$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = \ln(9)te^{2t}$</td>
<td>-1, -1</td>
<td>$(At + B)e^{2t}$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = e^{3t} \sin(t)$</td>
<td>-1, -1</td>
<td>$Ae^{3t} \cos(t) + Be^{3t} \sin(t)$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = e^{3t} \sin(t)$</td>
<td>-1, -1</td>
<td>$Ae^{3t} + B\cos(t) + C\sin(t)$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = e^{-t} \sin(t)$</td>
<td>-1, -1</td>
<td>$Ae^{-t} \cos(t) + Be^{-t} \sin(t)$</td>
</tr>
<tr>
<td>$y'' + 2y' + y = e^{-t} + \sin(t)$</td>
<td>-1, -1</td>
<td>$At^2 e^{-t} + B\cos(t) + C\sin(t)$</td>
</tr>
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Particular Solution General Form (Complex Roots):

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<tr>
<td>$y'' - 2y' + 2y = 4t^2 - 2$</td>
<td>$1 + i, 1 - i$</td>
<td>$A t^2 + B t + C$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = \sin(t)$</td>
<td>$1 + i, 1 - i$</td>
<td>$A \cos(t) + B \sin(t)$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = \sin(2t)$</td>
<td>$1 + i, 1 - i$</td>
<td>$A \cos(2t) + B \sin(2t)$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = \pi^2 e^{3t}$</td>
<td>$1 + i, 1 - i$</td>
<td>$A e^{3t}$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = 7te^{-t}$</td>
<td>$1 + i, 1 - i$</td>
<td>$(A t + B)e^{-t}$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = e^{3t} \sin(t)$</td>
<td>$1 + i, 1 - i$</td>
<td>$A e^{3t} \cos(t) + B e^{3t} \sin(t)$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = e^{3t} \cos(t)$</td>
<td>$1 + i, 1 - i$</td>
<td>$A e^{3t} \cos(t) + B e^{3t} \sin(t)$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = e^{t} \sin(t)$</td>
<td>$1 + i, 1 - i$</td>
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<td>$y'' - 2y' + 2y = e^{t} + \sin(t)$</td>
<td>$1 + i, 1 - i$</td>
<td>$A e^{t} + B \cos(t) + C \sin(t)$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = t e^{3t} \cos(t)$</td>
<td>$1 + i, 1 - i$</td>
<td>$(A t + B)e^{3t} \cos(t) + (C t + D)e^{3t} \sin(t)$</td>
</tr>
<tr>
<td>$y'' - 2y' + 2y = t e^{3t} \sin(t)$</td>
<td>$1 + i, 1 - i$</td>
<td>$(A t + B)t e^{t} \cos(t) + (C t + D)t e^{t} \sin(t)$</td>
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Solution algorithm:

1. Decide the correct general form for a particular solution $y_p(t)$ (with unknown coefficients) based on $f(t)$ in (1).
2. Differentiate $y_p(t)$ to get $y'_p(t)$ and $y''_p(t)$
3. Substitute $y_p(t)$, $y'_p(t)$, $y''_p(t)$ into (1).
4. By separating the equation into various components, calculate the unknown coefficients of $y_p(t)$.

Example (4.5.19). Find a general solution to the differential equation

$$y''(x) - 3y'(x) + 2y(x) = e^x \sin(x)$$

(4)
Solution: The hardest part is always deciding on the correct guess for the particular solution. Carefully matching each situation with cases 1 and 2 will take some of the guesswork out of the problems.

In this case the right hand side of (4) matches case 2 with:

\[ P_m(x) \equiv 0 \]
\[ Q_n(x) \equiv 1 \quad (\implies n = 0) \]
\[ \alpha = 1 \]
\[ \beta = 1 \]

There are only two unknown constants \((s \text{ and } k)\) that we need to work out before we can directly apply the formula:

\[
y_p(x) = x^s \left( A_k x^k + A_{k-1} x^{k-1} + \cdots + A_0 \right) e^{\alpha x} \cos(\beta x) \\
+ x^s \left( B_k x^k + B_{k-1} x^{k-1} + \cdots + B_0 \right) e^{\alpha x} \cos(\beta x) \quad (5)
\]

To decide whether \(s = 0\) or \(s = 1\) we solve the auxiliary equation \(r^2 - 3r + 2 = 0\). This has distinct roots \(r = 1, r = 2\) and since \(1 + i\) is not a root we have \(s = 0\) in (5).

The constant \(k\) is the maximum of the degrees of polynomials \(P_0\) and \(Q_0\). Since these are both constant polynomials (i.e. they both have degree 0) \(k = 0\).

Combining all of this we determine the correct trial function as given by:

\[
y_p(x) = A_0 e^x \cos(x) + B_0 e^x \sin(x) \quad (6)
\]

To determine \(A_0\) and \(B_0\) we need to substitute (6) into the differential equation (4), before we can do this we need to know \(y'_p(x)\) and \(y''_p(x)\).

Differentiating:

\[
y'_p(x) = A_0 (e^x \cos(x) - e^x \sin(x)) + B_0 (e^x \sin(x) + e^x \cos(x)) \\
= (A_0 + B_0) e^x \cos(x) + (-A_0 + B_0) e^x \sin(x)
\]

Twice:

\[
y''_p(x) = (A_0 + B_0) (e^x \cos(x) - e^x \sin(x)) + (-A_0 + B_0) (e^x \sin(x) + e^x \cos(x)) \\
= (2B_0) e^x \cos(x) + (-2A_0) e^x \sin(x)
\]
Substituting into (4) we get:

$$
\text{LHS} = y''(x) - 3y'(x) + 2y(x)
= [(2B_0)e^x \cos(x) + (-2A_0)e^x \sin(x)]
- 3 [(A_0 + B_0)e^x \cos(x) + (-A_0 + B_0)e^x \sin(x)]
+ 2 [A_0e^x \cos(x) + B_0e^x \sin(x)]
= [-B_0 - A_0] e^x \cos(x) + [A_0 - B_0] e^x \sin(x)
$$

$$
\text{RHS} = e^x \sin(x)
$$

Comparing we get:

$$
-B_0 - A_0 = 0
$$

$$
A_0 - B_0 = 1
$$

Solving for $A_0$ and $B_0$

$$
A_0 = \frac{1}{2}
$$

$$
B_0 = -\frac{1}{2}
$$

So a particular solution to (4) is given by

$$
y_p(x) = \frac{1}{2} e^x \cos(x) - \frac{1}{2} e^x \sin(x)
$$

(7)

The general solution to (4) is a sum of the general homogeneous solution $y_h(x)$ (determined by the auxiliary equation, method outlined here) and the particular solution $y_p(x)$ given by (7):

$$
y(x) = C_1e^x + C_2e^{2x} + \frac{1}{2} e^x \cos(x) - \frac{1}{2} e^x \sin(x)
$$

Example (4.5.24). Find the solution to the initial value problem

$$
y'' = 6t; \quad y(0) = 3, \quad y'(0) = -1
$$

(8)

Solution: First we find the (general) homogeneous solution.
The auxiliary equation is \( r^2 = 0 \) and so has the (repeated) root \( r = 0 \). This means the general homogeneous solution to (8) is given by

\[
y_h(t) = C_1e^{0t} + C_2te^{0t} = C_1 + C_2t
\]

In this case the right hand side of (8) matches case 1 with:

\[
P_m(t) = 6t \quad (\implies m = 1)
\]

\[\lambda = 0\]

Since \( \lambda = 0 \) is a repeated root of the auxiliary equation we have \( s = 2 \) in the formula for the particular solution to (8):

\[
y_p(t) = t^2 \left( A_1t^1 + A_0 \right) = A_1t^3 + A_0t^2
\]

Differentiating twice:

\[
y_p'(t) = 3A_1t^2 + 2A_0t
\]

\[
y_p''(t) = 6A_1t + 2A_0
\]

Substituting into (8) we get:

\[
\text{LHS} = y'' = 6A_1t + 2A_0
\]

\[
\text{RHS} = 6t
\]

Comparing we get:

\[
6A_1 = 6
\]

\[
2A_0 = 0
\]

Solving for \( A_0 \) and \( A_1 \):

\[
A_1 = 1
\]

\[
A_0 = 0
\]

So a particular solution to (8) is given by

\[
y_p(t) = t^3
\]

(9)
The general solution to (8) is

\[ y(t) = C_1 + C_2 t + t^3 \]

where we can solve for \( C_1 \) and \( C_2 \) by applying the initial conditions:

\[
\begin{align*}
y(0) = 3 & \implies C_1 = 3 \\
y'(0) = -1 & \implies C_2 = -1
\end{align*}
\]

Finally, this means that the solution to the IVP (8) is

\[ y(t) = 3 - t + t^3 \]