2nd Order Constant Coefficient Differential Equations (Non-Homogeneous Case)

Method of Undetermined Coefficients

The first method we learnt to tackle **non-homogeneous** constant coefficient differential equations is called the method of undetermined coefficients. This is a fancy name for what amounts to trial and error **used to determine the particular solution** y_p.

Consider the following differential equation:

$$ay'' + by' + cy = f(t) \tag{1}$$

The function $f(t)$ determines a guess for what the particular solution looks like (with some unknown coefficients). By substituting the guess into (1) we can solve for the coefficients and determine the $y_p(t)$ exactly.

It is important to note that **this method only works for specific functions** $f(t)$ that are made up of polynomials, exponentials and sinusoids (sines and cosines). For ease of notation let $P_m(t)$ and $Q_n(t)$ be polynomials of degree m and n respectively.

Case 1: $f(t) = P_m(t)e^{\lambda t}$

Use the form:

$$y_p(t) = t^s \left(A_m t^m + A_{m-1}t^{m-1} + \cdots + A_0\right) e^{\lambda t} \tag{2}$$

Where

$$s = \begin{cases} 0 & \text{if } \lambda \text{ is } \textbf{not a root} \text{ of the auxiliary equation} \\ 1 & \text{if } \lambda \text{ is } \textbf{a root} \text{ (not repeated) of the auxiliary equation} \\ 2 & \text{if } \lambda \text{ is } \textbf{a repeated root} \text{ of the auxiliary equation} \end{cases}$$

Case 2: $f(t) = P_m(t)e^{\alpha t}\cos(\beta t) + Q_n(t)e^{\alpha t}\sin(\beta t)$

Use the form:

$$y_p(t) = t^s \left(A_k t^k + A_{k-1} t^{k-1} + \cdots + A_0\right) e^{\alpha t} \cos(\beta t) + t^s \left(B_k t^k + B_{k-1} t^{k-1} + \cdots + B_0\right) e^{\alpha t} \sin(\beta t) \quad (3)$$

Where k is the maximum of m and n (the degrees of the polynomials $P_m(t)$ and $Q_n(t)$ respectively) and

$$s = \begin{cases} 0 & \text{if } \alpha + i\beta \text{ is } \textbf{not a root} \text{ of the auxiliary equation} \\ 1 & \text{if } \alpha + i\beta \text{ is } \textbf{a root} \text{ of the auxiliary equation} \end{cases}$$

Note: It is impossible for $\alpha + i\beta$ to be a repeated root (since $\alpha - i\beta$ is always the other root) so there is no option for $s = 2$.

Note: If you have a function $f(t)$ such as $\tan(t)$, $\ln(2t)$, $1/t$ etc, that is not given by any of the above you need to use variation of parameters instead to solve your problem.

Particular Solution General Form (Distinct Roots):

Differential Equation	Aux. roots	$y_p(t)$
$y'' - 3y' + 2y = 4t^2 - 2$	1,2	$At^2 + Bt + C$
$y'' - 3y' + 2y = \sin(t)$	1,2	$A\cos(t) + B\sin(t)$
$y'' - 3y' + 2y = \sin(2t)$	1,2	$A\cos(2t) + B\sin(2t)$
$y'' - 3y' + 2y = 12e^{3t}$	1,2	Ae^{3t}
$y'' - 3y' + 2y = 2e^{t}$	1,2	Ate^{t}
$y'' - 3y' + 2y = 4t^2e^{t}$	1,2	$(At^2 + Bt + C)te^{t}$
$y'' - 3y' + 2y = \frac{3}{4}te^{3t}$	1,2	$(At + B)e^{3t}$
$y'' - 3y' + 2y = t^3 - \cos(3t)$	1,2	$(At^3 + Bt^2 + Ct + D) + E\cos(3t) + F\sin(3t)$
$y'' - 3y' + 2y = t^2\cos(2t) + t^3\sin(2t)$	1,2	$(At^3 + Bt^2 + Ct + D)\cos(2t) + (Et^3 + Ft^2 + Gt + H)\sin(2t)$
$y'' - 3y' + 2y = e^{3t}\sin(t)$	1,2	$Ae^{3t}\cos(t) + Be^{3t}\sin(t)$
$y'' - 3y' + 2y = e^{3t} + \sin(t)$	1,2	$Ae^{3t} + B\cos(t) + C\sin(t)$
$y'' - 3y' + 2y = e^{t}\sin(t)$	1,2	$Ae^{t}\cos(t) + Be^{t}\sin(t)$
$y'' - 3y' + 2y = e^{t} + \sin(t)$	1,2	$Ate^{t} + B\cos(t) + C\sin(t)$

Particular Solution General Form (Repeated Root):

Differential Equation	Aux. roots	$y_p(t)$
$y'' + 2y' + y = 3t^2 + t$	-1, -1	$At^2 + Bt + C$
$y'' + 2y' + y = \sin(t)$	-1, -1	$A\cos(t) + B\sin(t)$
$y'' + 2y' + y = \sin(2t)$	-1, -1	$A\cos(2t) + B\sin(2t)$
$y'' + 2y' + y = \pi^2 e^{3t}$	-1, -1	Ae^{3t}
$y'' + 2y' + y = 5e^{-t}$	-1, -1	At^2e^{-t}
$y'' + 2y' + y = 7te^{-t}$	-1, -1	$(At + B)t^2e^{-t}$
$y'' + 2y' + y = \ln(9)te^{2t}$	-1, -1	$(At + B)e^{2t}$
$y'' + 2y' + y = e^{3t}\sin(t)$	-1, -1	$Ae^{3t}\cos(t) + Be^{3t}\sin(t)$
$y'' + 2y' + y = e^{3t} + \sin(t)$	-1, -1	$Ae^{3t} + B\cos(t) + C\sin(t)$
$y'' + 2y' + y = e^{-t}\sin(t)$	-1, -1	$Ae^{-t}\cos(t) + Be^{-t}\sin(t)$
$y'' + 2y' + y = e^{-t} + \sin(t)$	-1, -1	$At^2e^{-t} + B\cos(t) + C\sin(t)$

Particular Solution General Form (Complex Roots):

Differential Equation	Aux. roots	$y_p(t)$
$y'' - 2y' + 2y = 4t^2 - 2$	$1+i,\ 1-i$	$At^2 + Bt + C$
$y'' - 2y' + 2y = \sin(t)$	$1+i,\ 1-i$	$A\cos(t) + B\sin(t)$
$y'' - 2y' + 2y = \sin(2t)$	$1+i,\ 1-i$	$A\cos(2t) + B\sin(2t)$
$y'' - 2y' + 2y = \pi^2 e^{3t}$	$1+i,\ 1-i$	Ae^{3t}
$y'' - 2y' + 2y = 7te^{-t}$	$1+i,\ 1-i$	$(At+B)e^{-t}$
$y'' - 2y' + 2y = e^{3t}\sin(t)$	$1+i,\ 1-i$	$Ae^{3t}\cos(t) + Be^{3t}\sin(t)$
$y'' - 2y' + 2y = e^{3t} + \cos(t)$	$1+i,\ 1-i$	$Ae^{3t} + B\cos(t) + C\sin(t)$
$y'' - 2y' + 2y = e^{t}\sin(t)$	$1+i,\ 1-i$	$Ate^{t}\cos(t) + Bte^{t}\sin(t)$
$y'' - 2y' + 2y = e^{t}\sin(2t)$	$1+i,\ 1-i$	$Ae^{t}\cos(2t) + Be^{t}\sin(2t)$
$y'' - 2y' + 2y = e^{2t}\sin(t)$	$1+i,\ 1-i$	$Ae^{2t}\cos(t) + Be^{2t}\sin(t)$
$y'' - 2y' + 2y = e^{t} + \sin(t)$	$1+i,\ 1-i$	$Ae^{t} + B\cos(t) + C\sin(t)$
$y'' - 2y' + 2y = te^{3t}\cos(t)$	$1+i,\ 1-i$	$(At+B)e^{3t}\cos(t) + (Ct+D)e^{3t}\sin(t)$
$y'' - 2y' + 2y = te^{t}\sin(t)$	$1+i,\ 1-i$	$(At+B)te^{t}\cos(t) + (Ct+D)te^{t}\sin(t)$

Solution algorithm:

1. Decide the correct general form for a particular solution $y_p(t)$ (with unknown coefficients) based on $f(t)$ in (1).
2. Differentiate $y_p(t)$ to get $y_p'(t)$ and $y_p''(t)$
3. Substitute $y_p(t)$, $y_p'(t)$, $y_p''(t)$ into (1).
4. By separating the equation into various components, calculate the unknown coefficients of $y_p(t)$.

Example (4.5.19)**.** *Find a general solution to the differential equation*

$$y''(x) - 3y'(x) + 2y(x) = e^{x}\sin(x) \tag{4}$$

Solution: The hardest part is always deciding on the correct guess for the particular solution. Carefully matching each situation with cases 1 and 2 will take some of the guesswork out of the problems.

In this case the right hand side of (4) matches case 2 with:

$$\begin{aligned} P_m(x) &\equiv 0 \\ Q_n(x) &\equiv 1 \; (\implies n = 0) \\ \alpha &= 1 \\ \beta &= 1 \end{aligned}$$

There are only two unknown constants (s and k) that we need to work out before we can directly apply the formula:

$$\begin{aligned} y_p(x) = x^s &\left(A_k x^k + A_{k-1}x^{k-1} + \cdots + A_0\right) e^{\alpha x}\cos(\beta x) \\ &+ x^s \left(B_k x^k + B_{k-1}x^{k-1} + \cdots + B_0\right) e^{\alpha x}\cos(\beta x) \end{aligned} \tag{5}$$

To decide whether $s = 0$ or $s = 1$ we solve the auxiliary equation $r^2 - 3r + 2 = 0$. This has distinct roots $r = 1$, $r = 2$ and since $1 + i$ is **not** a root we have $s = 0$ in (5).

The constant k is the maximum of the degrees of polynomials P_0 and Q_0. Since these are both constant polynomials (i.e. they both have degree 0) $k = 0$.

Combining all of this we determine the correct trial function as given by:

$$y_p(x) = A_0 e^x \cos(x) + B_0 e^x \sin(x) \tag{6}$$

To determine A_0 and B_0 we need to substitute (6) into the differential equation (4), before we can do this we need to know $y_p'(x)$ and $y_p''(x)$.

Differentiating:

$$\begin{aligned} y_p'(x) &= A_0\left(e^x\cos(x) - e^x\sin(x)\right) + B_0\left(e^x\sin(x) + e^x\cos(x)\right) \\ &= (A_0 + B_0)e^x\cos(x) + (-A_0 + B_0)e^x\sin(x) \end{aligned}$$

Twice:

$$\begin{aligned} y_p''(x) &= (A_0 + B_0)\left(e^x\cos(x) - e^x\sin(x)\right) + (-A_0 + B_0)\left(e^x\sin(x) + e^x\cos(x)\right) \\ &= (2B_0)e^x\cos(x) + (-2A_0)e^x\sin(x) \end{aligned}$$

Substituting into (4) we get:

$$\begin{aligned}
\text{LHS} &= y''(x) - 3y'(x) + 2y(x) \\
&= [(2B_0)e^x\cos(x) + (-2A_0)e^x\sin(x)] \\
&\quad - 3\left[(A_0+B_0)e^x\cos(x) + (-A_0+B_0)e^x\sin(x)\right] \\
&\quad + 2\left[A_0e^x\cos(x) + B_0e^x\sin(x)\right] \\
&= [-B_0 - A_0]\, e^x\cos(x) + [A_0 - B_0]\, e^x\sin(x) \\
\text{RHS} &= e^x\sin(x)
\end{aligned}$$

Comparing we get:

$$\begin{aligned}
-B_0 - A_0 &= 0 \\
A_0 - B_0 &= 1
\end{aligned}$$

Solving for A_0 and B_0

$$\begin{aligned}
A_0 &= \frac{1}{2} \\
B_0 &= -\frac{1}{2}
\end{aligned}$$

□

So a particular solution to (4) is given by

$$y_p(x) = \frac{1}{2}e^x\cos(x) - \frac{1}{2}e^x\sin(x) \tag{7}$$

The general solution to (4) is a sum of the general homogeneous solution $y_h(x)$ (determined by the auxiliary equation, method outlined here) and the particular solution $y_p(x)$ given by (7):

$$y(x) = C_1e^x + C_2e^{2x} + \frac{1}{2}e^x\cos(x) - \frac{1}{2}e^x\sin(x)$$

Example (4.5.24)**.** *Find the solution to the initial value problem*

$$y'' = 6t; \quad y(0) = 3, \quad y'(0) = -1 \tag{8}$$

Solution: First we find the (general) homogeneous solution.

The auxiliary equation is $r^2 = 0$ and so has the (repeated) root $r = 0$. This means the general homogeneous solution to (8) is given by

$$\begin{aligned} y_h(t) &= C_1 e^{0t} + C_2 t e^{0t} \\ &= C_1 + C_2 t \end{aligned}$$

In this case the right hand side of (8) matches case 1 with:

$$\begin{aligned} P_m(t) &= 6t \ (\implies m = 1) \\ \lambda &= 0 \end{aligned}$$

Since $\lambda = 0$ is a repeated root of the auxiliary equation we have $s = 2$ in the formula for the particular solution to (8):

$$y_p(t) = t^2 \left(A_1 t^1 + A_0 \right) = A_1 t^3 + A_0 t^2$$

Differentiating twice:

$$\begin{aligned} y_p'(t) &= 3A_1 t^2 + 2A_0 t \\ y_p''(t) &= 6A_1 t + 2A_0 \end{aligned}$$

Substituting into (8) we get:

$$\begin{aligned} \text{LHS} &= y'' \\ &= 6A_1 t + 2A_0 \\ \text{RHS} &= 6t \end{aligned}$$

Comparing we get:

$$\begin{aligned} 6A_1 &= 6 \\ 2A_0 &= 0 \end{aligned}$$

Solving for A_0 and A_1:

$$\begin{aligned} A_1 &= 1 \\ A_0 &= 0 \end{aligned}$$

So a particular solution to (8) is given by

$$y_p(t) = t^3 \tag{9}$$

The general solution to (8) is

$$y(t) = C_1 + C_2 t + t^3$$

where we can solve for C_1 and C_2 by applying the initial conditions:

$$\begin{aligned} y(0) = 3 &\implies C_1 = 3 \\ y'(0) = -1 &\implies C_2 = -1 \end{aligned}$$

Finally, this means that the solution to the IVP (8) is

$$y(t) = 3 - t + t^3$$

□