2nd Order Constant Coefficient Differential Equations (Non-Homogeneous Case)

Variation of Parameters

To solve second order differential equations

$$ay'' + by' + cy = f(t) \tag{1}$$

where $f(t)$ is a general (continuous) function we can use **variation of parameters**. This method is generally more complicated than the Method of Undetermined Coefficients so you would typically want to use it in problems when you cannot apply the method of undetermined coefficients.

Note: This method **requires** us to already have two linearly independent solutions $y_1(t)$ and $y_2(t)$ to the corresponding homogeneous equation

$$ay'' + by' + cy = 0 \tag{2}$$

These solutions will either be given to you, or you will be able to construct these yourself[1]

Solution Algorithm:

1. Determine two linearly independent solutions $y_1(t)$, $y_2(t)$ to (2)
2. Set up the system of equations:

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= \frac{f}{a} \end{aligned} \tag{3}$$

3. Solve for v_1 and v_2:

$$\begin{aligned} v_1'(t) &= \frac{-f(t) y_2(t)}{a[y_1(t) y_2'(t) - y_1'(t) y_2(t)]} \\ v_2'(t) &= \frac{f(t) y_1(t)}{a[y_1(t) y_2'(t) - y_1'(t) y_2(t)]} \end{aligned} \tag{4}$$

[1] We have learnt to do this in two cases: When (1) has constant coefficients a, b, c and when (1) is a Cauchy-Euler equation (Section 4.7) with variable coefficients.

Note: It is up to you but I would not recommend memorizing the formulae above. Instead it is an easier and more consistent strategy to solve the system of equations (3) by hand each time. This way if you make a mistake you will at least get partial credit for your method!

4. Integrate $v_1'(t)$ and $v_2'(t)$ to determine $v_1(t)$ and $v_2(t)$.
5. A particular solution y_p to (1) is given by

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

Example (4.6.1)**.** *Find a general solution to the differential equation using the method of variation of parameters.*

$$y'' + 4y = \tan(2t) \tag{5}$$

Note: The question specifically asked us to use variation of parameters but even without this prompt we would have been forced to do so because $\tan(2t)$ is not a function we can apply the method of undetermined coefficients to.

Solution: We need to first find two linearly independent solutions to the homogeneous equation corresponding to (5):

$$y'' + 4y = 0$$

We've solved many of these constant coefficient problems in sections 4.2 and 4.3, method is summarized in the handout: Constant Coefficient Homogeneous Differential Equations

Solving the auxiliary equation $r^2 + 4 = 0$ determines two (linearly independent) homogeneous solutions:

$$y_1(t) = \cos(2t)$$
$$y_2(t) = \sin(2t)$$

This means a particular solution will be given by:

$$y_p(t) = v_1(t)\cos(2t) + v_2(t)\sin(2t)$$

To determine v_1 and v_2 we set up a system of equations as in (3):

$$\begin{aligned} \cos(2t)v_1'(t) + \sin(2t)v_2'(t) &= 0 \\ -2\sin(2t)v_1'(t) + 2\cos(2t)v_2'(t) &= \frac{\tan(2t)}{1} \end{aligned} \tag{6}$$

Let's first calculate $v_1(t)$.

Multiplying the top equation of (6) by $2\cos(2t)$ and the bottom equation by $\sin(2t)$ we get:

$$\begin{aligned} 2\cos^2(2t)v_1'(t) + 2\cos(2t)\sin(2t)v_2'(t) &= 0 \\ -2\sin^2(2t)v_1'(t) + 2\cos(2t)\sin(2t)v_2'(t) &= \tan(2t)\sin(2t) \end{aligned}$$

Subtract:

$$-2(\cos^2(2t) + \sin^2(2t))v_1'(t) = \tan(2t)\sin(2t) \tag{7}$$

Since $\cos^2(2t) + \sin^2(2t) = 1$ for all t we can simplify (7) as:

$$-2v_1'(t) = \tan(2t)\sin(2t)$$

So:

$$v_1'(t) = -\frac{1}{2}\tan(2t)\sin(2t) \implies v_1(t) = -\frac{1}{2}\int \tan(2t)\sin(2t)$$

This is a little tricky to calculate since it is not a common trigonometric integral but it can be rewritten as one since

$$\tan(2t)\sin(2t) = \frac{\sin^2(2t)}{\cos(2t)}$$

So:

$$\begin{aligned} v_1(t) = -\frac{1}{2}\int \tan(2t)\sin(2t)dt &= -\frac{1}{2}\int \frac{\sin^2(2t)}{\cos(2t)}dt \\ &= -\frac{1}{2}\int \frac{1-\cos^2(2t)}{\cos(2t)}dt \\ &= -\frac{1}{2}\int \sec(2t) - \cos(2t)dt \\ &= -\frac{1}{4}\ln|\tan(2t) + \sec(2t)| + \frac{1}{4}\sin(2t) \end{aligned}$$

Similarly we can rearrange (6) to solve for $v_2'(t)$:

$$v_2'(t) = \frac{1}{2}\sin(2t) \implies v_2 = \frac{1}{2}\int \sin(2t)dt = -\frac{1}{4}\cos(2t)$$

So finally we get a particular solution to (5)

$$\begin{aligned} y_p(t) &= \frac{1}{4}\left(\sin(2t) - \ln|\tan(2t) + \sec(2t)|\right)\cos(2t) - \frac{1}{4}\sin(2t)\left(\cos(2t)\right) \\ &= -\frac{1}{4}\ln|\tan(2t) + \sec(2t)| \end{aligned}$$

The general solution to (5) is a sum of the general homogeneous solution and a particular solution so here it is given by:

$$y(t) = c_1\cos(2t) + c_2\sin(2t) - \frac{1}{4}\ln|\tan(2t) + \sec(2t)|$$

$\square$

Example (4.6.7)**.** *Find a general solution to the differential equation using the method of variation of parameters.*

$$y'' + 4y' + 4y = e^{-2t}\ln(t) \tag{8}$$

Solution: As before we begin by finding two linearly independent solutions to the homogeneous equation corresponding to (8). In this case the auxiliary equation is $r^2 + 4r + 4 = 0$ with repeated solution $r = -2$ which determines two linearly independent solutions:

$$\begin{aligned} y_1(t) &= e^{-2t} \\ y_1(t) &= te^{-2t} \end{aligned}$$

This means a particular solution will be given by:

$$y_p(t) = v_1(t)e^{-2t} + v_2(t)te^{-2t}$$

Set up a system of equations:

$$\begin{aligned} e^{-2t}v_1'(t) + te^{-2t}v_2'(t) &= 0 \\ -2e^{-2t}v_1'(t) + (e^{-2t} - 2te^{-2t})v_2'(t) &= \frac{e^{-2t}\ln(t)}{1} \end{aligned} \tag{9}$$

To simplify cancel e^{-2t} (we are allowed to divide by e^{-2t} here because it is never 0) to get:

$$\begin{aligned} v_1'(t) + tv_2'(t) &= 0 \\ -2v_1'(t) + (1-2t)v_2'(t) &= \ln(t) \end{aligned} \tag{10}$$

It seems easier here to first solve for v_2' by multiplying the top equation by -2 to get:

$$\begin{aligned} -2v_1'(t) - 2tv_2'(t) &= 0 \\ -2v_1'(t) + (1-2t)v_2'(t) &= \ln(t) \end{aligned}$$

Subtract and integrate:

$$v_2'(t) = \ln(t) \implies v_2(t) = \int \ln(t)dt$$

Evaluating this requires an integration by parts trick with $u = \ln(t)$ and $v' = 1$. Applying the integration by parts formula we get:

$$\int \ln(t)dt = t\ln(t) - \int t \cdot \frac{1}{t}dt = t(\ln(t) - 1)$$

Rearranging the top equation of (10) we get:

$$v_1'(t) = -tv_2'(t) = -t(\ln(t)) \implies v_1(t) = \int -t\ln(t)dt$$

To evaluate this we can use integration by parts with $u = \ln(t)$ and $v' = -t$, applying the integration by parts formula:

$$\begin{aligned} v_1(t) = \int -t\ln(t)dt &= \left(-\frac{1}{2}t^2\right)\ln(t) - \int \left(-\frac{1}{2}t^2\right) \cdot \frac{1}{t}dt \\ &= -\frac{1}{2}t^2\ln(t) + \frac{1}{2}\int tdt \\ &= -\frac{1}{2}t^2\ln(t) + \frac{1}{4}t^2 \end{aligned}$$

So a particular solution to (8) is given by:

$$
\begin{aligned}
y_p(t) &= \frac{1}{4}t^2(1-\ln(t))e^{-2t} + t(\ln(t)-1)te^{-2t} \\
&= \frac{1}{4}t^2e^{-2t} - \frac{1}{4}t^2e^{-2t}\ln(t) + t^2e^{-2t}\ln(t) - t^2e^{-2t} \\
&= -\frac{3}{4}t^2e^{-2t} + \frac{1}{2}t^2e^{-2t}\ln(t) \\
&= \frac{1}{4}t^2e^{-2t}(2\ln(t)-3)
\end{aligned}
$$

The general solution to (8) is then given by:

$$y(t) = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{4}t^2e^{-2t}(2\ln(t)-3)$$

□