Instructions

1. Write your Name and PID on the front of your Blue Book.
2. No calculators or other electronic devices are allowed during this exam.
3. You may use a double sided page of notes.
4. Write your solutions clearly in your Blue Book.
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order as they appear in the exam.
   (c) Start each numbered problem on a new side of a page.
5. Show all of your work and justify all your claims. No credit will be given for unsupported answers, even if correct.

Complete 5 out of the 6 questions

1. (10 points) Find the general solution to the differential equation

   \[ \frac{dy}{dx} = \frac{y}{x} + x^2 \cos(x) \]

   **Proof.** This is a linear differential equation.

   Rearrange into standard form

   \[ \frac{dy}{dx} + \left( -\frac{1}{x} \right) y = x^2 \cos(x) \]

   Calculate integrating factor:

   \[ \mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln(x)} = \frac{1}{x} \]

   Multiply by \( \mu \) and simplify:

   \[ \frac{d}{dx} \left( \frac{1}{x} \cdot y \right) = x \cos(x) \]

   Integrate

   \[ \frac{1}{x} \cdot y = \int x \cos(x) dx + C \]

   \[ = x \sin(x) + \cos(x) + C \]

   General solution:

   \[ y = x(x \sin(x) + \cos(x) + C) \]

2. (10 points) Solve the initial value problem

   \[ \frac{dy}{dx} = (x + y)^2 - (x - y)^2, \quad y(1) = e^2 \]
Proof. Once you simplify the right hand side it becomes clear that this is a separable equation:

\[ \frac{dy}{dx} = 4xy \]

Separate variables and integrate:

\[ \int \frac{1}{y} \, dy = \int 4x + C \]

Calculating we get:

\[ \ln |y| = 2x^2 + C \]

We are given that \( y > 0 \) so we simplify \( \ln |y| = \ln(y) \) and can write the solution as:

\[ y = Ae^{2x^2} \]

Applying the initial value we get:

\[ e^2 = Ae^2 \implies A = 1 \]

So the (explicit) solution is given by:

\[ y = e^{2x^2} \]

3. (10 points) [This question has multiple parts]

(a) Find the general solution \( y_h(t) \) to the homogeneous differential equation

\[ \frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 9y = 0 \]

(b) Give the general form of a particular solution \( y_p(t) \) to

\[ \frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 9y = te^t \] \quad (1)

(You do not need to solve for any unknown constants)

(c) Using (a) and (b) give the general solution to the non-homogeneous equation (1)

Proof. (a) The auxiliary equation is given by

\[ r^2 + 6r + 9 = 0 \]

which has \( r = -3 \) as a repeated root. So the general form of the homogeneous solution is given by:

\[ y_h(t) = C_1 e^{-3t} + C_2 te^{-3t} \]

(b) Since \( r = +1 \) is not a root to the auxiliary equation the general form of the particular solution is given by

\[ y_p(t) = (At + B)e^t \]
(c) The general solution is a sum of the homogeneous and particular solution:

\[ y(t) = y_h(t) + y_p(t) = C_1e^{-3t} + C_2te^{-3t} + (At + B)e^t \]

4. (10 points) Find the general solution to the differential equation

\[ \frac{dy}{dx} = \frac{2x - y}{x + y - 4} \]

**Hint:** It may be a good idea to rewrite this as an equation involving a differential form.

**Proof.** Rewriting we get:

\[ (2x - y)dx - (x + y - 4)dy = 0 \]

This determines:

\[ M = 2x - y, \quad N = -(x + y - 4) \]

Verify exactness:

\[ M_y = -1, \quad N_x = -1 \]

Integrating \( M \) with respect to \( x \) we get:

\[ F(x, y) = x^2 - xy + g(y) \quad (2) \]

Differentiate with respect to \( y \) and compare with \( N \) to get:

\[ -x + g'(y) = -x - y + 4 \Rightarrow g'(y) = -y + 4 \]

Integrate \( g'(y) \) and substitute into (2):

\[ F(x, y) = x^2 - xy - \frac{1}{2}y^2 + 4y \quad (3) \]

The solutions are given (implicitly) by:

\[ x^2 - xy - \frac{1}{2}y^2 + 4y = C \]

For some constants \( C \)

5. (10 points) Solve the initial value problem

\[ y''(t) + 4y(t) = 4 \sin(2t); \quad y(0) = 1, \quad y'(0) = 3 \quad (4) \]

**Hint:** The general form of the particular solution to (4) is given by

\[ y_p(t) = At \cos(2t) \]
Proof. The auxiliary equation is given by
\[ r^2 + 4 = 0 \]
and so has two complex roots \(+2i\) and \(-2i\). So the general homogeneous solution is given by:
\[ y_h(t) = C_1 \cos(2t) + C_2 \sin(2t) \]
Using the hint we calculate:
\[ y'_p = A \cos(2t) - 2At \sin(2t) \]
\[ y''_p = -2A \sin(2t) - 2A(\sin(2t) + 2t \cos(2t)) \]
Substituting into the differential equation we get:
\[ -4A = 4 \implies A = -1 \]
The general solution then has the form:
\[ y(t) = C_1 \cos(2t) + C_2 \sin(2t) - t \cos(2t) \]
To apply the initial conditions we need to know \(y'(t)\) so differentiate:
\[ y'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t) - (\cos(2t) - 2t \sin(2t)) \]
Apply initial conditions at \(t = 0\) to get:
\[ C_1 = 1 \]
\[ 2C_2 - 1 = 3 \implies C_2 = 2 \]

6. (10 points) Solve the initial value problem for the **Cauchy-Euler equation**
\[ t^2 y''(t) + 7t y'(t) + 5y(t) = 0; \quad y(1) = -1, \quad y'(1) = 13 \]
Proof. The auxiliary equation is given by
\[ r^2 + 6r + 5 = 0 \]
and has distinct roots \(r = -1, -5\). The general form of the homogeneous solution is then given by:
\[ y(t) = C_1 t^{-1} + C_2 t^{-5} \]
To apply the initial conditions we need to know \(y'(t)\) so differentiate:
\[ y'(t) = -C_1 t^{-2} - 5C_2 t^{-6} \]
Apply initial conditions at \(t = 1\) to get:
\[ C_1 + C_2 = -1 - C_1 - 5C_2 = 13 \]
So we get:
\[ C_1 = 2 \]
\[ C_2 = -3 \]