Separable Equations

The first non-trivial examples of differential equations we will learn to solve are called separable. The main idea in solving these equations is to separate variables then integrate respectively.

Definition:

A first-order differential equation of the form

\[ \frac{dy}{dx} = f(x, y) \]

is called separable if the right hand side function \( f(x, y) \) can be written as a product

\[ f(x, y) = g(x) h(y) \]

where \( g \) and \( h \) are functions of \( x \) only and \( y \) only respectively.

Some examples of separable and non-separable equations (by no means an exhaustive list):

<table>
<thead>
<tr>
<th>Separable</th>
<th>Non-separable</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dx}{dt} = x^2 t^2 )</td>
<td>( \frac{dx}{dt} = x^2 + t^2 )</td>
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<tr>
<td>( \frac{dx}{dt} = \frac{yt}{\sqrt{t^2 - 1}} )</td>
<td>( \frac{dx}{dt} = \frac{yt}{\sqrt{t^2 - 1}} )</td>
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<tr>
<td>( \frac{dy}{dx} = e^{2x+3y} )</td>
<td>( \frac{dy}{dx} = \log(2x + 3y) )</td>
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<tr>
<td>( \frac{dr}{d\theta} = \frac{r(e^\theta + 3)}{5r\theta} )</td>
<td>( \frac{dr}{d\theta} = \frac{3(e^\theta + r)}{5r\theta} )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = xy + y + x + 1 )</td>
<td>( \frac{dy}{dx} = xy + y + x )</td>
</tr>
</tbody>
</table>

Solution algorithm:

1. Rewrite your differential equation into the following separated form (if necessary):

\[ \frac{dy}{dx} = g(x) \cdot h(y) \tag{1} \]

2. Multiply (1) by \( dx \) and divide by \( h(y) \) to get

\[ \frac{dy}{h(y)} = g(x) \cdot dx \tag{2} \]
3. The (implicit) solution is given by integrating (2) (\textbf{don’t forget} \( +C \) on the right hand side)

\[ \int \frac{dy}{h(y)} = \int g(x) \cdot dx + C \]

4. If given an initial condition, solve for \( C \)

**Example** (2.2.13). Solve the equation

\[
\frac{dy}{dx} = 3x^2(1 + y^2)^{3/2} \tag{3}
\]

*Solution:* First we verify that this is a separable equation. This is easily seen to be the case with \( g(x) = 3x^2 \) and \( h(y) = (1 + y^2)^{3/2} \)

Rearranging (3) we get:

\[
\frac{dy}{(1 + y^2)^{3/2}} = 3x^2 \, dx
\]

Integrate both sides:

\[
\int \frac{dy}{(1 + y^2)^{3/2}} = \int 3x^2 \, dx + C \tag{4}
\]

To evaluate (4) we need to calculate the left hand side which is a tricky integral. Fortunately it can be made easier with a substitution. Applying the substitution \( y = \tan(u) \) we transform the integral in \( y \) into an integral in \( u \) in the following way:

\[
\int \frac{dy}{(1 + y^2)^{3/2}} \longmapsto \int \frac{du \cdot \sec^2(u)}{\sec^3(u)}
\]

Simplifying:

\[
\int \frac{du \cdot \sec^2(u)}{\sec^3(u)} = \int \cos(u) du = \sin(u)
\]

We need to rewrite this as a function of \( y \). Since \( y = \tan(u) \) we have \( u = \arctan(y) \), and so:

\[
\sin(u) = \sin(\arctan(y))
\]
We can go much further in simplifying this complicated trigonometric function. Notice that:

\[ y = \tan(u) \implies y^2 = \tan^2(u) = \sec^2(u) - 1 \]

Rearranging we get:

\[ \sec^2(u) = y^2 + 1 \]

So:

\[ \cos^2(u) = \frac{1}{y^2 + 1} \quad (5) \]

Sine and cosine are related by the formula:

\[ \sin^2(u) + \cos^2(u) = 1 \quad (6) \]

Using (5) and (6) we get:

\[
\sin(u) = \sqrt{1 - \cos^2(u)} \\
= \sqrt{\frac{y^2}{y^2 + 1}} \\
= \sqrt{1 - \frac{1}{y^2 + 1}} \\
= \frac{y}{\sqrt{y^2 + 1}}
\]

This lets us finally calculate the left hand side of (4) to get the (implicit) solution.

\[
\frac{y}{\sqrt{y^2 + 1}} - x^3 = C \quad (7)
\]

(We have no given initial condition so we cannot solve for C)

Example (2.2.22). Solve the initial value problem

\[ x^2dx + 2ydy = 0, \quad y(0) = 2 \quad (8) \]

Solution: This problem is not stated in the standard \( \frac{dy}{dx} = f(x, y) \) form, but notice that it is still separated (in fact this is more of less the equation we expect at the end of step 2). We rearrange:

\[ 2ydy = -x^2dx \]
and integrate:
\[ \int 2y\,dy = \int -x^2\,dx + C \]
Calculating we get:
\[ y^2 = -\frac{1}{3}x^3 + C \]
Using the initial condition we can calculate \( C \) by substituting \( x = 0, \, y = 2 \)
\[ (2)^2 = -\frac{1}{3}(0)^3 + C \implies C = 4 \]
The (implicit) solution to (8) is then given by:
\[ y^2 = 4 - \frac{1}{3}x^3 \quad (9) \]
Going further we can take the square root:
\[ y = \pm \sqrt{4 - \frac{1}{3}x^3} \]
The initial condition \( y(0) = 2 \) forces us to take the positive square root here to get the explicit solution:
\[ y = \sqrt{4 - \frac{1}{3}x^3} \quad (10) \]