Handout on Continuity

Ross M. Richardson

Continuity is a very general concept, extending well beyond analysis.[1] As such, it should not be surprising that one can define and use continuity in a number of ways.

1 The Three-Fold Way

We shall suppose in what follows that X and Y are metric spaces with corresponding metrics d_X and d_Y. We then have the following definition:

Definition 1 ($\varepsilon - \delta$ Definition). *If $E \subseteq X$, $p \in E$ and $f : E \to Y$, then f is* continuous *at p if for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$d_Y(f(x), f(p)) < \varepsilon$$

for every point $x \in E$ for which $d_X(x, p) < \delta$.

Note that the definition of continuity is point-wise. If f is continuous at every point of E then f is continuous on E.

We have thus far seen a number of similar definitions involving little greek letters–remember the above definition applies to a function between two metric spaces. While this definition certainly looks the most familiar, it is not always the most convenient to work with. In particular, there are two distinct and equivalent definitions of continuity which are easier to use and, miraculously, involve no greek letters!

Definition 2 (Sequence Definition). *Given the notation above, and the assumption p is a limit point, f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$. Moreover, f is continuous at p if and only if for every sequence $p_n \to p$ we have $\lim_{n\to\infty} f(p_n) = f(p)$.*

This theorem says that continuity is exactly the condition that allows you to "bring limits inside" in the sense that $\lim_{n\to\infty} f(p_n) = f(\lim_{n\to\infty} p_n) = f(p) = q$ so long as f is defined at p (the first definition allows for the possibility that f is not defined at p, but this is a small issue). For instance, on the previous homework you were asked (indirectly) to show that $\lim_{n\to\infty} \sqrt{1 + 1/n} = 1$. This takes a little work if you work from the definition of convergence. However, if you know that the square root function is continuous, then the fact that $\{1 + 1/n\}_{n=1}^{\infty}$ converges to 1 implies that

$$\lim_{n\to\infty} \sqrt{1 + 1/n} = \sqrt{\lim_{n\to\infty} 1 + 1/n} = \sqrt{1} = 1.$$

[1] In particular, the concept of continuity is fundamental for *Topology*, which is offered at UCSD as Math 190.

Additionally, note that we have banished the little greek letters. Where did they go? Well, the assumption that $\sqrt{\cdot}$ was a continuous function allowed us to sweep them under the rug. This makes the sequence definition very powerful.

There is yet a third definition of continuity! We have

Definition 3 (Open Set Definition). *A mapping $f : X \to Y$ is continuous if and only if for every open set V the set*

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

is open. The same is true if we replace open with closed.

Observe that once again, there are no ϵs in sight! Moreover, this notion of continuity is not pointwise. Indeed, this definition deals with neither sequences nor greek letters, but rather with open and closed sets. It says that the *inverse image* of an open (closed) set is open (closed). While not immediately intuitive, this can be very useful when proving facts about open and closed sets. When you work on next week's homework, consider carefully which definition is easiest to use (hint: rarely is the $\varepsilon - \delta$ definition easiest).

Finally, why can I say that all of the above are *definitions* for continuity? The reason is that all of the above are equivalent, that is, if you have one of the definitions the others hold as well. Most of these are proved in Rudin, but it is a good exercise to try to prove these implications. Come see me for help.

2 Open, Closed, Connected, Compact

We saw that if E is an open set, then $f^{-1}(E)$ is an open set. What about $f(E)$? Consider the map $f : [0, 1] \to [0, 1]$ which sends every element to 0 (i.e. $f(x) = 0, 0 \leq x \leq 1$). The image set $f([0, 1]) = \{0\}$ is *not* open. Indeed, we have a number of conditions that a set E in a metric space can satisfy. Try to fill out the following table indicating whether each property is necessarily preserved under forward or inverse image. For example, we have that if E is open then $f^{-1}(E)$ is open but $f(E)$ is not necessarily open. For every "No",

	$f(E)$	$f^{-1}(E)$
E open	No	Yes
E closed		
E compact		
E connected		

find a counterexample. For every "Yes", try to find/discover the proof. You will learn a great deal about how continuity works by doing this. We will discuss the answers next week.