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Math 262 Lectures Notes

Eigenvalues for Random Graphs

1 Eigenvalues for graphs

The eigenvalues associated with a graph can give us a lot of information. In this lecture we will start working towards the topic for finding the eigenvalues for power law graphs.

In the previous two lectures we got a bound for the largest eigenvalue of a graph. In [1] it is shown that in the case that $\sqrt{m}$ is much larger than $\tilde{d}$ that $\lambda_k = (1 + o(1))\sqrt{m_k}$ where m_k is the kth largest expected degree, given that m_k is sufficiently large. This implies that under certain conditions that a random graph in the $G(w)$ model where w has a power law distribution will have its eigenvalues distributed with a power law distribution.

Another distribution of eigenvalues that arises frequently is the one developed by Eugene Wigner and is known as the semi-circle law. With this distribution we have that most of the eigenvalues are tightly around some central point, the name semi-circle comes by noting the density of eigenvalues forms a semi-circle shape.

$$\text{Power law distribution} \quad \longleftrightarrow \quad \text{Semi-circle law distribution}$$

These two distributions appear inconsistent with each other. But both arise when studying the eigenvalues of a random graph. The difference is from what matrix we get the eigenvalues from.

1.1 Matrices associated with a graph

There are a variety of ways to associate a matrix to a graph. The different matrices will yield different information about the structure of the graph.

The obvious association is the **adjacency matrix** A where the entries are determined by the following rule.

$$a_{ij} = (A)_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \text{ is an edge,} \\ 0 & \text{if } \{i,j\} \text{ is not an edge.} \end{cases}$$

Last time we saw that the largest eigenvalue of the adjacency matrix measures the capacity of the graph.

We have also seen the **combinatorial Laplacian** $L = D - A$ (where D is the diagonal matrix whose entries are the degrees of the vertex and A is the adjacency matrix) when we outlined the proof of the matrix-tree theorem. Note that this matrix is symmetric (since both D and A are), singular (since the all 1's vector is an eigenvector associated with the eigenvalue 0) and positive semi-definite (most easily seen by noting that since it is symmetric all of the eigenvalues are real and then by the Geršgorin disc theorem the eigenvalues must all be nonnegative forcing the matrix to be positive semi-definite).

Closely related is the **normalized Laplacian** $\mathcal{L} = D^{-1/2}LD^{-1/2} = I - D^{-1/2}AD^{-1/2}$. By construction it has many of the same properties as L, namely it is a symmetrical (more particularly, self-adjoint), singular, positive semi-definite matrix. This matrix was introduced in [2] to give a symmetric matrix that contains information about a random walk.

A random walk on a matrix is formed by looking at how an initial distribution on the vertices evolves over time. To describe the behavior of a random walk we let

$$P(u,v) = Pr(\text{Going from } u \text{ to } v) = \begin{cases} \frac{1}{d_u} & \text{if } u \text{ adjacent to } v, \\ 0 & \text{else.} \end{cases}$$

(In a more general model we can weight the edges so that when u is adjacent to v we have $P(u,v) = \frac{w_{uv}}{d_u}$, but we will not go into this at this time.) Now letting $f = [f(v)]$ be a row vector representing an initial distribution on the vertices of the graph then after one step in our random walk our distribution will become $fP(v) = \sum_{u \in G} f(u)P(u,v)$. This can be achieved by multiplying f on the *left* by the matrix $P = D^{-1}A$, but the matrix P is not symmetric and so we do not have any control over its eigenvalues, hence the reason to look for a matrix such as $\mathcal{L}$ for which we have more control.

1.2 Normalized Laplacian eigenvalues

Since $\mathcal{L}$ is a singular, positive semi-definite matrix then the eigenvalues of $\mathcal{L}$ can be written as

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}.$$

The size of the gap between λ_0 and λ_1 gives us information about the Cheeger constant which tells us about the connectivity, or the shape of a graph. This relates to the isoperimitic equality related to the following constant of a graph,

$$\min_A \left(\frac{E(A,\overline{A})}{\min\{vol(A), vol(\overline{A})\}} \right)$$

where we let A range over all nontrivial subsets of the vertices of the graph, $E(A,\overline{A})$ denotes the number of edges connecting vertices in A to those in $\overline{A} = V(G) \setminus A$ and $vol(A) = \sum_{i \in A} d_i$. Intuitively this is a measure of the minimal amount of cutting needed to disconnect the graph. In the case of a random graph it is very hard to make a cut, stated otherwise, it is hard to get a small bottleneck inside of a random graph. (It can be shown that $\lambda_1 = 0$ if and only if the graph is disconnected.)

What we want to show is that $\max_{i \neq 0} |1 - \lambda_i|$ is small. That is all of the eigenvalues except the zero eigenvalue will cluster around 1.

2 Eigenvalues of $G(w)$

We would like to get some control on the eigenvalues of the normalized Laplacian of a random graph G from our $G(w)$ model. Recall that $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$ where A is the adjacency matrix where each entry is a random indicator variable where $a_{ij} = 1$ with probability $w_i w_j \rho$ and 0 else. Similarly, D is the diagonal matrix where $d_i = \sum_j a_{ij}$ which is a sum of independent random variables (we have that $E(d_i) = w_i$ and by use of the concentration inequalities we have some control on d_i).

Since $\mathcal{L}$ is a real symmetric matrix then by the spectral decomposition theorem we know that there exists a set of n orthonormal eigenvectors, $x_0, \ldots, x_{n-1}$ with $\mathcal{L}x_i = \lambda_i x_i$. We can use these eigenvectors to construct n projection matrices P_i where P_i projects onto the vector x_i, namely $P_i = x_i x_i^*$. By construction we have $P_i^2 = P_i$ (since x_i has unit length), for $i \neq j$ we have $P_i P_j = 0$ (since x_i is orthogonal to x_j), and $I = \sum_{i=0}^{n-1} P_i$ (since the x_i form an orthonormal basis).

Similarly, $\mathcal{L} = \sum_{i=0}^{n-1} \lambda_i P_i$ as can be seen by noting that $\big(\sum_{i=0}^{n-1} \lambda_i P_i\big)x_j = \lambda_j x_j = \mathcal{L}x_j$ and so as a linear transformation both A and $\sum_{i=0}^{n-1} \lambda_i P_i$ agree on a basis so they must be equal. Using the rules outlined above for P_i we have that $\mathcal{L}^k = \sum_{i=0}^{n-1} \lambda_i^k P_i$. Note that this construction is equivalent to saying that $\mathcal{L}$ is unitarily similar to a diagonal matrix.

We want to show that all of the eigenvalues except zero cluster around 1. This is equivalent to bounding the spectral radius of the matrix

$$M = I - \mathcal{L} - P_0 = \sum_{i \neq 0} (1 - \lambda_i) P_i,$$

which has eigenvalues $0, 1 - \lambda_1, \ldots, 1 - \lambda_{n-1}$, is small which will bound the maximum value of $|1 - \lambda_i|$. Working back we see that

$$M = D^{-1/2}AD^{-1/2} - \phi_0 \phi_0^*,$$

where D and A are defined as before and ϕ_0 is a column vector whose ith entry is $d_i / \sqrt{\sum_i d_i}$. The problem with working with this form is that the resulting M will have a large number of random variables multiplying together and combining in nontrivial ways.

So alternatively we want to consider

$$C = W^{-1/2}AW^{-1/2} - \chi\chi^*,$$

where A is again the adjacency matrix but now W is the diagonal matrix with diagonal entries w_i (which are fixed) and χ is a column vector whose ith entry is $w_i / \sqrt{\sum_i w_i}$ (which again are fixed). So C involves linear combinations of random variables, making it easier to work with and control.

It remains to argue that C is a good approximation for M and then find a bound for C. This will be done in the next lecture.

References

[1] Chung, F., Lu L. and Vu, V. ”Eigenvalues of random power law graphs”, Annals of Combinatorics **7** 21-33, 2003

[2] Chung, F. *Lectures on Spectral Graph Theory*, AMS, 1997