Packing equal squares into a large square

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Abstract

Let $s(x)$ denote the maximum number of non-overlapping unit squares which can be packed into a large square of side length x. Let $W(x) = x^2 - s(x)$ denote the "wasted" area, i.e., the area not covered by the unit squares. In this note we prove that

$$W(x) = O\big(x^{(3+\sqrt{2})/7}\log x\big).$$

This improves earlier results of Erdős-Graham and Montgomery in which the upper bounds of $W(x) = O(x^{7/11})$ and $W(x) = O(x^{(3-\sqrt{3})/2}\log x)$, respectively, were obtained. A complementary problem is to determine $s'(x)$ the minimum number of unit squares needed to cover a large square of side length x. We show that

$$s'(x) = x^2 + O\big(x^{(3+\sqrt{2})/7}\log x\big),$$

improving an earlier bound of $x^2 + O(x^{7/11})$.

1 Introduction

The problem of finding dense packings of equal squares into a square has developed a fairly substantial literature since it was first introduced some 35 years ago in a paper of Erdős and the second author [1]. Most of the research has centered on the case when the number of squares to be packed is relatively small, e.g., less than 100. (The reader can consult Freedman [3] for an excellent survey of the current state of knowledge.)

In this note we show that a square of side length x can be packed with unit squares so that the *uncovered* (or *waste*) area $W(x)$ satisfies

$$W(x) = O\big(x^{(3+\sqrt{2})/7}\log x\big).$$

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The best previous bound $W(x) = O(x^{(3-\sqrt{3})/2})$ was due to Montgomery [8] (see also [9]), which improved on the earlier result in Erdős and Graham [1] that $W(x) = O(x^{7/11})$. Note that

$$\begin{aligned} 7/11 &= 0.636363\ldots, \\ \frac{3-\sqrt{3}}{2} &= 0.633974\ldots, \\ \frac{3+\sqrt{2}}{7} &= 0.630601\ldots. \end{aligned}$$

By results of Roth and Vaughan [9], it is known that $W(x) \gg x^{1/2-\epsilon}$ for any $\epsilon > 0$ when x is bounded away from an integer. More precisely they show that if $x(x - \lfloor x \rfloor) > 1/6$, then

$$W(x) > 10^{-100}\sqrt{x\,|x - \lfloor x + 1/2 \rfloor|}.$$

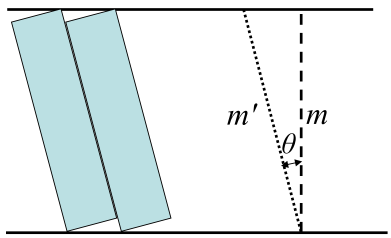
Needless to say, their proof is non-trivial.

The method of proving our new bound on $W(x)$ can also be used to improve the bounds for the dual problem of *covering* a square of side length x by a minimum number of unit squares. Previously, Karabash and Soifer [5] showed that the total number $s'(x)$ of squares needed satisfies $s'(x) = x^2 + O(x^{2/3})$. Recently they improved the bound to $s'(x) = x^2 + O(x^{7/11})$ (see [6, 10]), based on the earlier estimate of $W(x) = O(x^{7/11})$. We will further improve this bound to

$$s'(x) = x^2 + O\big(x^{(3+\sqrt{2})/7}\log x\big).$$

2 Preliminaries

During this and subsequent sections we will frequently suppress lower order terms in our estimates for ease of exposition. Suppose we have a strip of width m. (Here m is a function of x that goes to infinity as x approaches infinity.) We wish to pack a stack of unit squares of height $\lceil m \rceil$ as close to being orthogonal as possible. As seen in Figure 1, the stack will form an angle θ which is no more than $\sqrt{2/m}$.



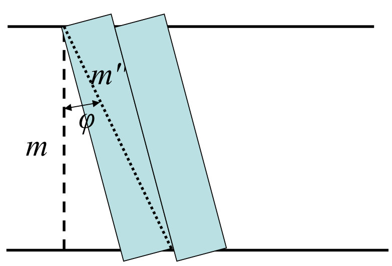
$$m' = \lceil m \rceil + \tan\theta$$

$$f = m' - m$$

$$\theta = \arctan\frac{\sqrt{(m')^2 - m^2}}{m} \leq \arctan\frac{\sqrt{2mf + f^2}}{m} \approx \sqrt{\frac{2}{m}}$$

Figure 1: Packing a stack of squares.

Suppose now we wish to cover a strip of width m by using stacks of unit squares of height $\lceil m \rceil + 1$ as in Figure 2. Again the stacks form an angle φ which is no more than $2/\sqrt{m}$ (plus lower order terms). Note that φ is larger than θ but is of the same order.



$$m'' = \sqrt{\lceil m \rceil^2 + 1}$$

$$f' = \lceil m \rceil - m$$

$$\varphi = \arctan\frac{\sqrt{(m'')^2 - m^2}}{m} \leq \arctan\frac{\sqrt{2m(f'+1) + (f'+1)^2}}{m} \approx \frac{2}{\sqrt{m}}$$

Figure 2: Covering with a stack of squares.

3 The construction

The proof will be an induction based on efficient packings of three basic shapes:

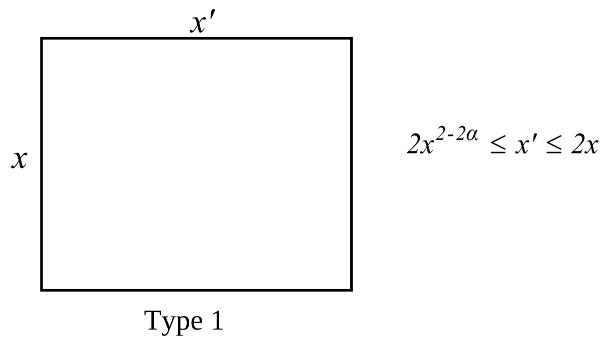


Figure 3: Type 1 rectangle.

A Type 1 shape is a rectangle with a vertical side length x and base x' with $2x^{2-2\alpha} \leq x' \leq 2x$ where $\alpha = (3+\sqrt{2})/7$ (see Figure 3).
A Type 2 shape is a trapezoid with a vertical left side length of x, a top edge of length x' satisfying $\sqrt{x} \leq x' \leq 2x$ and the angle θ between the right-hand side and a vertical line satisfying $\theta \leq \sqrt{2/x}$ (see Figure 4).
A Type 3 shape is also a trapezoid similar to Type 2, except that in this case the top edge length x' satisfies $2x^{2-2\gamma} \leq x' \leq 2x$ with $\gamma = 1/\sqrt{2}$, and the angle θ satisfies $\theta \leq 1/x$ (see Figure 5). Let W_i, $1 \leq i \leq 3$, denote the minimum

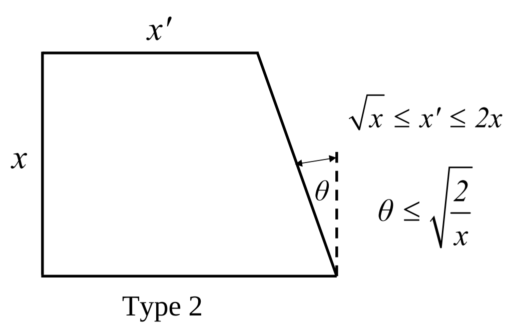


Figure 4: Type 2 trapezoid.

possible waste when a Type i shape is packed with unit squares.

Theorem 1 *For a suitable absolute constant $c > 0$,*

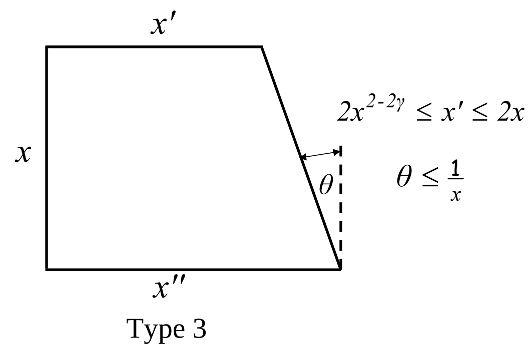


Figure 5: Type 3 trapezoid.

(1)

$$W_1(x) \leq 4cx^{\alpha} \log x \quad \textit{where } \alpha = \frac{3+\sqrt{2}}{7};$$

(2)

$$W_2(x) \leq cx^{\beta} \log x \quad \textit{where } \beta = \frac{2+\sqrt{2}}{4};$$

(3)

$$W_3(x) \leq 2cx^{\gamma} \log x \quad \textit{where } \gamma = \frac{1}{\sqrt{2}}.$$

Note that $W(x) \leq W_1(x) = O(x^{\alpha} \log x)$.

Proof of (1):

We will tile our x by x' Type 1 rectangle R as follows. We first pack an integer-sided rectangle $R' \subseteq R$ perfectly with unit squares (i.e., with no waste), leaving unfilled two strips: S_1 of width m and length x, and S_2 of width m' and length $x' - m$, where $m, m' \approx x^{2-2\alpha}$ (see Figure 6). The strips in S and S' consists of stacks of unit squares of lengths $\lceil m \rceil$ and $\lceil m' \rceil$, respectively. The four unfilled regions at each of the ends of S and S' are (essentially) Type 2 trapezoids, with $\theta \leq \sqrt{2/m}$. The wasted (i.e., uncovered) area along the border of S and S' is bounded by $4\sqrt{2}x/\sqrt{m}$. Hence, the waste in this packing of R is

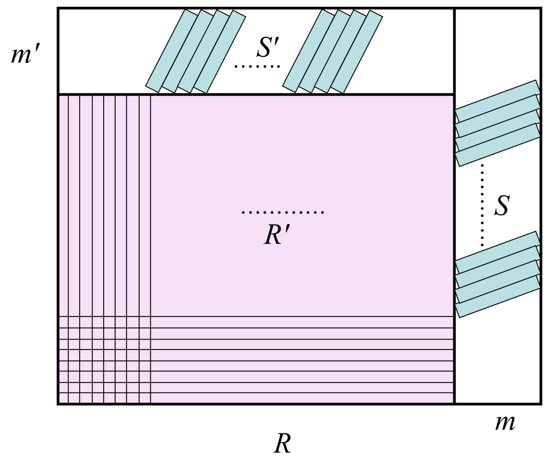


Figure 6: Packing of Type 1.

(by induction) bounded above by

$$\begin{aligned}4\sqrt{2}\frac{x}{\sqrt{m}} + 4W_2(m) &\leq 6x^{\alpha} + 4cm^{\beta}\log m \\ &\leq 6x^{\alpha} + 4cx^{(2-2\alpha)\beta}(\log x^{2-2\alpha}) \\ &\leq 4cx^{\alpha}\log x\end{aligned}$$

since $6 + 4c(2-2\alpha)\log x \leq 4c\log x$ for $x \geq 2$ and $c \geq 10$.

Proof of (2):

Our plan is to partition the Type 2 trapezoid T_2 into a perfectly packed region A, and about $\sqrt{2x}$ of the Type 3 shapes of side length about $\sqrt{x/2}$, together with a strip S at the bottom of (approximate) dimensions $x^{2-2\alpha}$ by x' (see Figure 7). (Thus, S is a Type 1 shape.) Also note that with $y = \sqrt{x/2}$, each B_i is a Type 3 shape (since the corresponding angle $\theta \leq 1/\sqrt{x} = 1/y$ and B_i has side length $y = \sqrt{x}$). Hence, by induction, the total waste for this packing is bounded above by

$$(2x)^{\alpha} + \sqrt{2x}c(\sqrt{x/2})^{\gamma}\log\sqrt{x/2} \leq cx^{\beta}\log x$$

since $(1+\gamma)/2 \leq \beta$ and c is large enough.

Proof of (3):

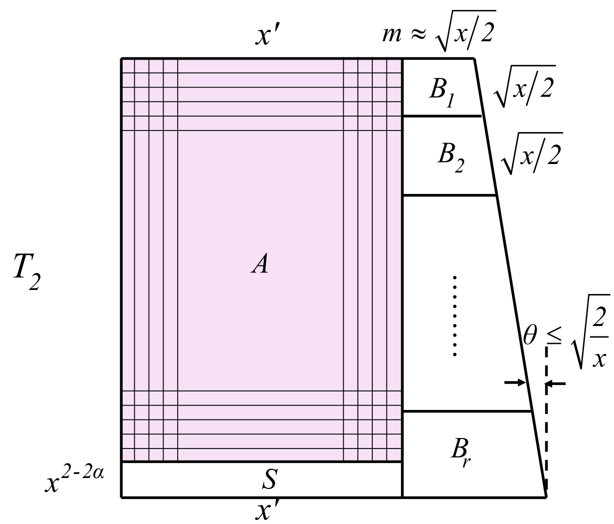


Figure 7: Packing of Type 2.

We will have to work a little harder for this case. In Figure 8 we show a Type 3 shape T_3. We are going to partition T_3 into various regions (similar to what was done for Type 2). As before, the bulk of T_3 consists of an integer-sided rectangle A' which will be packed perfectly. Note that the top and bottom edge lengths x' and x'' differ by at most 1 since $\theta \leq 1/x$.

For each C_i, we are going to pack most of it with tilted stacks of unit squares of height $z = 1 + \lceil w \rceil$ (which is one more than what we could use, but which will be useful for our purpose). In particular, such a stack can be used for all the C_i (i.e., it is longer than the bottom edge of the last C_i).

Now, for each C_i we partition it into a triangular region e_i and a rectangular region R_i (see Figure 9). The sum of the areas of e_i (which will all be wasted) is at most

$$\frac{x}{k} \cdot \frac{k^2}{x} = k \leq x^\gamma \qquad \text{since} \;\; \theta \leq \frac{1}{x}.$$

The rectangular region R_i will be packed with tilted stacks of z unit squares, but leaving spaces at the end (to be specified later). The total wasted space

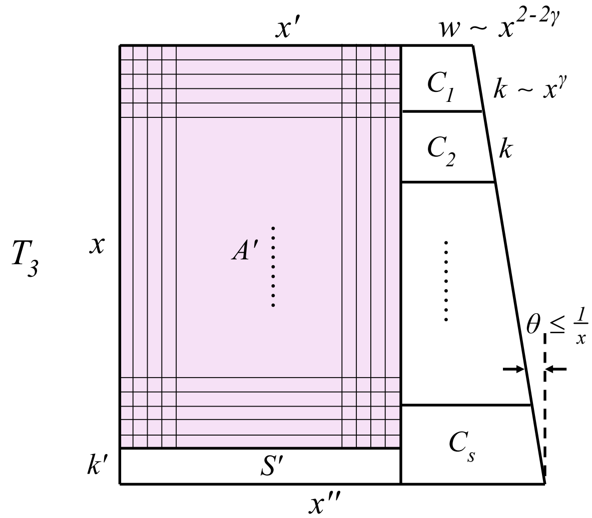


Figure 8: Packing of Type 3.

along the borders of the C_i is bounded by

$$\frac{2k\sqrt{2}}{\sqrt{w}} \cdot \frac{x}{k} = \frac{2\sqrt{2}k}{\sqrt{w}} \leq 3x^{\gamma}.$$

Now comes a more subtle step. We are going to examine the transition between C_i and C_{i+1} (see Figure 10). The plan is that at each such transition we will stop short of the dividing line between C_i and C_{i+1} by about $w^{2-2\gamma}/2$ and form a new trapezoidal shape with the union of the two ends (see Figure 11) by trimming off small triangular pieces. It will turn out to be a Type 3 shape because the difference of the angles σ' and σ'' is sufficiently small (as we will soon compute). At the top of C_1 and the bottom of C_s we leave a gap of length $w^{2-2\gamma}$, so that Type 2 shapes are formed. By induction, the waste here is at most

$$2 \cdot 4cw^{\beta} \log w \leq x^{\gamma}.$$

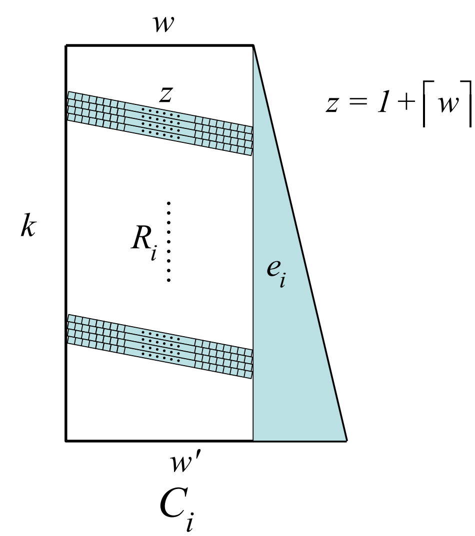


Figure 9: Packing of C_i.

In particular, the difference of the angles σ_i and σ_{i+1} is bounded by

$$\begin{aligned}
\sigma_i - \sigma_{i+1} &= 2\sqrt{\frac{\delta_i}{z}} - 2\sqrt{\frac{\delta_{i+1}}{z}} \qquad \text{where } \delta_i = z - w_i \\
&\le 2\sqrt{\frac{\delta_i}{z}} - 2\sqrt{\frac{\delta_i - x^{-1+\gamma}}{z}} \qquad \text{since } w_{i+1} \le w_i + x^{-1+\gamma} \\
&= 2\sqrt{\frac{\delta_i}{z}}\Big(1 - \sqrt{1 - \frac{x^{-1+\gamma}}{\delta_i}}\Big) \\
&\le 2\sqrt{\frac{\delta_i}{z}} \cdot \frac{1}{2} \cdot \frac{x^{-1+\gamma}}{\delta_i} \\
&\le \frac{1}{\sqrt{\delta_i} z} \\
&\le \frac{1}{z}
\end{aligned}$$

since $z \approx w \approx x^{2-2\gamma}$. This shows that the "trimmed" shape between C_i and C_{i+1} is a Type 3 trapezoid and so by the induction hypothesis, the total waste for those pieces is bounded above by

$$\begin{aligned}
x^{1-\gamma} w^{\gamma} &\le x^{1-\gamma+(2-2\gamma)\gamma} c \log x^{2-2\gamma} \\
&\le \frac{3}{5} c x^{\gamma} \log x.
\end{aligned}$$

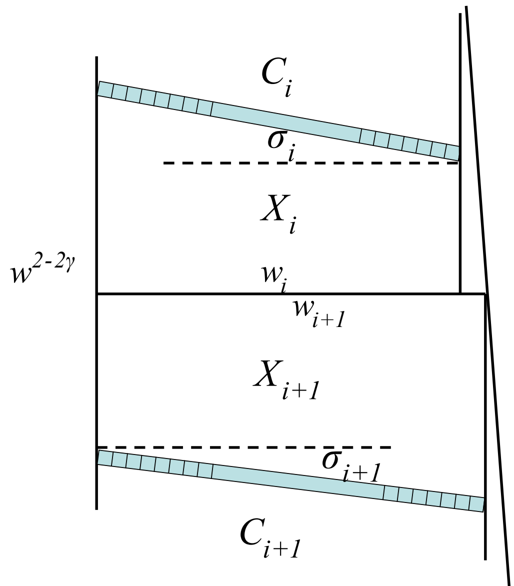


Figure 10: Transition between C_i and C_{i+1}.

We must also bound the waste due to the trimming used in creating these transition Type 3 shapes. However it is easy to see that this waste is bounded above by

$$\begin{aligned} \frac{2x}{k} w^{2-2\gamma} &\leq 2x^{1-\gamma+(2-2\gamma)^2} \\ &\leq x^{\gamma}. \end{aligned}$$

Finally, we must bound the waste due the strip S' at the bottom. Here it is immediate that this waste is at most

$$cx^{\alpha} \log x + w^{\gamma} \leq x^{\gamma}$$

since S' is essentially the union of a rectangle and a Type 3 trapezoid. Putting this altogether, we see that the total waste is no more than

$$8x^{\gamma} + \frac{3}{5} cx^{\gamma} \log x < cx^{\gamma} \log x.$$

This completes the induction step. We will choose c sufficiently large so that

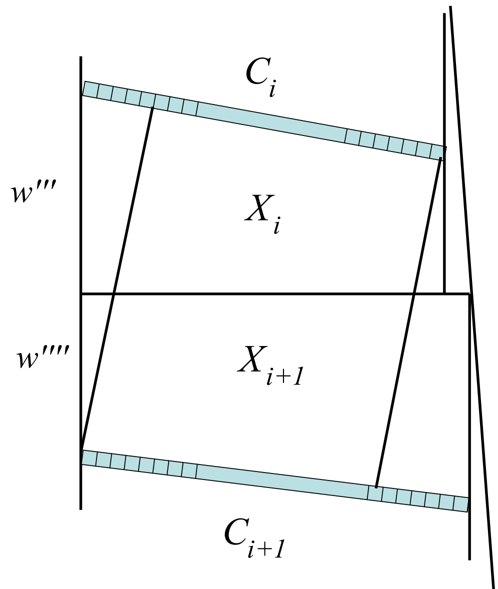


Figure 11: Packing between C_i and C_{i+1}.

for small x, say $x \leq 100$, the theorem holds. This establishes the initial step of the induction and the proof is complete. □

4 Some variations

Instead of packing unit squares into a large square, we consider covering a large square with unit squares. We can prove the following:

Theorem 2 *We can cover a square of side length x using $s'(x)$ unit squares with*

$$s'(x) = x^2 + O(x^\alpha \log x) \quad \textit{where} \quad \alpha = \frac{3+\sqrt{2}}{7}.$$

The proof proceeds along the same lines as described in the preceding section using simultaneous induction on Types 1, 2 and 3 shapes. The only difference is that the angle φ in the Type 2 trapezoids is a constant factor larger than the associated angle θ computed in Section 2. The inductive proofs are almost identical to those of Theorem 1. (We omit the proof.)

Let W_θ denote the waste for packing unit squares in trapezoid with a general angle θ. A Type 2 shape is the special case with $\theta = \sqrt{2/x}$ and a Type 3 shape is the case that $\theta = 1/x$. It is not difficult to use the same proof to show:

Theorem 3 *We can pack unit squares into a trapezoid of side length x and angle θ so that the waste $W_\theta(x)$ satisfies*

$$W_\theta(x) \quad = \quad \begin{cases} O(x^{1-\delta/(2+\sqrt{2})}) & \text{if} \quad \theta = 1/x^\delta \quad \text{and} \quad 0 < \delta \le 2\alpha, \\ O(x^\alpha) & \text{if} \quad \theta \le 1/x^{2\alpha}. \end{cases}$$

This proof is very similar to the proof of Theorem 1 and we will not include the proof here.

5 Concluding remarks

A natural question to ask at this point is to what extent the exponent bound of $(3+\sqrt{2})/7$ can be improved.

Challenge ($100). Improve the bound $(3+\sqrt{2})/7$ to $(3+\sqrt{2})/7 - c$ for some $c > 0$.

While it might be natural to think that the "truth" is $W(x) = O(x^{1/2+\epsilon})$ for any $\epsilon > 0$ (as was suggested in [1]), the authors are skeptical. In fact, we offer the opposite:

Conjecture: ($1000). Prove that for some $c > 0$, $W(x) \gg x^{1/2+c}$.

(Of course, this reward will be paid for a disproof of the conjecture.) As usual, such prizes are only given to the first valid claimant!

The upper bound on $W(x)$ can be used to improve a related bound on $t(N)$, defined to be the edge length of the smallest square into which n non-overlapping unit squares can be packed (cf [7]). For this problem, we obtain

$$t(N) \le N^{1/2} + O(\frac{\log N}{N^{(4-\sqrt{2})/14}}),$$

improving earlier estimates of the big-O term based on weaker estimates of $W(x)$.

Many questions remain. For example, if we have a trapezoid shape similar to Type 2 but with an angle $\theta > c > 0$ for some positive c, then is it true that the waste area is greater than $c'x$ for some constant $c' = c'(\theta) > 0$?

In the other direction, if the angle of a trapezoid is small, say $\theta < 1/x^{2\alpha}$, it can be shown that the wasted area is $O(x^\alpha)$ as stated in Theorem 3. It would be of interest to find the maximum θ such that the trapezoid has waste area of the same order as a square.

In this paper, we deal with both the wasted area of packing and covering a large square by unit squares. Are the two quantities $W(x)$ and $C(x) = s'(x) - x^2$ of the same order (as suggested in [6])?

Suppose that, for an integer n, we wish to pack $n^2 + 1$ unit squares into a large square with side length, say $n + \delta_n$, as small as possible. What is the values of δ_n? The only known values are $\delta_1 = 1$ and $\delta_2 = 1/\sqrt{2}$ (see [3]). In [7] it was shown that $\delta_{43} \leq 1/2$. An immediate consequence of Theorem 1 is

$$\delta_n \ll n^{(-5+\sqrt{2})/7},$$

improving an earlier estimate [7] using [1]. On the other hand, it seems likely to us that the smallest square into which $n^2 - 1$ unit squares can be packed has side length n. The same result should hold for packing $n^2 - k$ unit squares when k is fixed and n is sufficiently large (as conjectured in [3]).

By way of contrast, the problem of packing equal discs into a large equilateral triangle seems to exhibit a somewhat different behavior. Let $T(m)$ denote the smallest possible side length of an equilateral triangle into which m non-overlapping unit discs can be packed. It is known [2] that $T\left(\binom{n+1}{2}\right) = n - 1 + 2\sqrt{3}$. Two currently unproved conjectures [4] are:

Conjecture: For some constant $c > 0$, $T\left(\binom{n+1}{2} + 1\right) > T\left(\binom{n+1}{2}\right) + c$ for all n. In fact, computation suggests that $c > 0.3$.

Conjecture: $T\left(\binom{n+1}{2} - 1\right) = T\left(\binom{n+1}{2}\right)$ for all n.

In other words, the optimal (obvious) packing of a triangular number of equal discs into an equilateral triangle is so good that no smaller triangle can hold one fewer disc, and further, if we try to pack one more, then the triangle side length must increase by some non-trivial positive amount. However, it is easy to see that a smaller triangle can be used if we are packing two fewer discs.

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