Open problems in Euclidean Ramsey Theory

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Abstract

Ramsey theory is a branch of combinatorics that deals with unavoidable structure in large systems. Euclidean Ramsey theory specifically looks at structure that remains when some geometric object is partitioned, and this note is a survey of some open problems in that field.

1 Introduction

Ramsey theory is the study of structure that must exist in a system, most typically after it has been partitioned. A good example is the well-known theorem of van der Waerden [28], which states that given any $k, r \in \mathbb{N}$, there exists a least integer $w(k, r)$ such that if $[w(k, r)] := \{1, \ldots, w(k, r)\}$ is partitioned into r sets (or r-*colored*), then there exists a monochromatic arithmetic progression of length k (that is, a k-term arithmetic progression in one of the sets).

Euclidean Ramsey theorems are similar in nature, but are concerned with geometric objects – most often $\mathbb{E}^n$ or partitions of graphs with geometric properties, such as the hypercube embedded in $\mathbb{E}^n$. Euclidean Ramsey theory abounds with open problems, nearly all of them elementary to state. First, though, we need some definitions.

For a finite set $X \subset \mathbb{E}^k$, let *Cong(X)* denote the set of all subsets of $\mathbb{E}^k$ which are *congruent* to X under some Euclidean motion. We will say that X is *Ramsey* if for every integer r, there is a least integer $N(X, r)$ such that if $N \geqslant N(X, r)$ then for any r-coloring of $\mathbb{E}^N$, there is a monochromatic $X' \in Cong(X)$. We denote this property by the usual "arrow" notation $\mathbb{E}^N \to X$. The negation of this statement is denoted by $\mathbb{E}^N \not\to X$.

It is not hard to see that any Ramsey set must be finite. Furthermore, it follows from compactness arguments (where we are using the Axiom of Choice) that if X is Ramsey, then in fact there must be a *finite* set S such that $S \to X$.

A more restricted notion is that of being r-Ramsey. This just means that a monochromatic copy of X must occur whenever the underlying set S is r-colored. In this case, we write $S \xrightarrow{r} X$. The negation of this statement is written as $S \stackrel{r}{\not\to} X$.

Conjecture 1 ([9]). *For any non-equilateral triangle T, (i.e., the set of 3 vertices of T),*

$$\mathbb{E}^2 \xrightarrow{2} T.$$

Conjecture 2. *For any triangle T, there exists a 3-coloring of $\mathbb{E}^2$ without a monochromatic copy of T.*

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For any particular equilateral triangle T, one can color the plane with alternating half-open red and blue strips with height the altitude of T, avoiding a monochromatic copy of T. This was conjectured to be essentially the only possible 2-coloring that avoids any triangle, though recently V. Jelínek, J Kynčl, R. Stolař and T. Valla showed that there exist infinitely many such colorings [13]. They have also shown that Conjecture 1 is true if one color class is open and the other is closed.

Conjecture 1 is known to be true for many classes of triangles; a partial list can be found in [9], and a proof that it is true for right triangles due to L. Shader is in [22]. Less seems to be known about Conjecture 2; the 3-coloring by alternating half-open strips avoids a large class of triangles, but not all. Recently we have discovered a 3-coloring of $\mathbb{E}^2$ that avoids the degenerate triangle with sides $a, a, 2a$, shown in Figure 1. This tiling extends to cover $\mathbb{E}^2$; each hexagon has diameter $2a$ and all of the hexagons are half-open as shown for the uppermost hexagon in Figure 1.

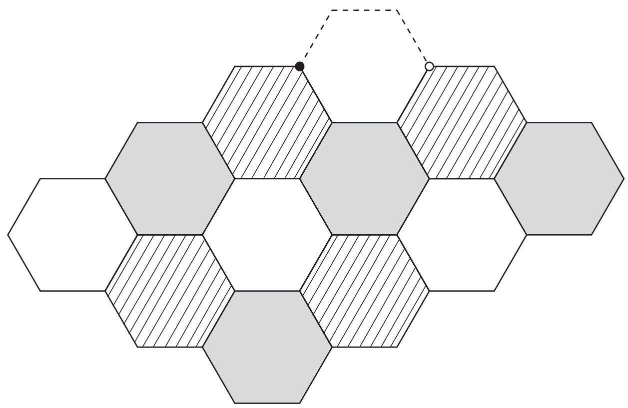

Figure 1: A sketch of the 3-coloring avoiding the $(a, a, 2a)$ triangle.

For any collinear set S, it is known that with 16 colors one can avoid a monochromatic copy of L in $\mathbb{E}^n$ for all n [27], but it is an open question if this is the best possible.

2 Ramsey Sets

It is a long-standing problem to discover which sets are Ramsey, as defined above. Igor Křiž has some strong positive theorems to this end:

Theorem 1 ([14]). *Suppose $X \subseteq \mathbb{E}^N$ has a transitive group of isometries with a solvable subgroup with at most two orbits. Then X is Ramsey.*

Theorem 2 ([15]). *If X is the set of vertices of a trapezoid, X is Ramsey.*

Frankl and Rödl [10] have shown that every non-degenerate simplex is Ramsey. It is shown in [8] that any Ramsey set must lie on the surface of a sphere (in some dimension) – we call such sets *spherical*. In particular, collinear sets, as pointed out above, can always be avoided with at most 16 colors in any number of dimensions. It may turn out that the Ramsey sets are very easy to describe:

Conjecture 3. *(\$1000). Every spherical set is Ramsey.*

A weaker conjecture is:

Conjecture 4. *(\$100). Every* 4*-point subset of a circle is Ramsey.*

Recall that given a Ramsey set X and an integer r, we defined $N(X,r)$ to be the least integer such that if $N \geqslant N(X,r)$ then for any r-coloring of $\mathbb{E}^N$, there is a monochromatic $X' \in Cong(X)$. One might go farther than asking which sets are Ramsey; given a Ramsey set and an integer r, what can we say about $N(X,r)$? As we'll see below, even for the simplest nontrivial case, a 2-point set, this is a major open question.

3 Unit Distance Graphs

A *unit distance graph* in a metric space (X,ρ) is a graph $G=(X,E)$ with vertex set X and edge set $\{x,y \in X : \rho(x,y)=1\}$. $\|\cdot\|$ will denote the usual Euclidean norm. $\chi(X,\rho)$ will denote the chromatic number of X under the metric ρ, though we omit ρ when it is the Euclidean norm.

The most widely-known problem in Euclidean Ramsey theory is probably that of determining the chromatic number of the plane, $\chi(\mathbb{E}^2)$. This question is attributed to Nelson (see [23], [24], and [25] for a full account), and there is a wide literature surrounding it. Despite its broad interest, the best known bounds are $4 \leq \chi(\mathbb{E}^2) \leq 7$. Proof of the lower bound is in Figure 2 – it is a unit distance graph with chromatic number 4, usually known as the Moser spindle, after Leo Moser. The upper bound is given by a hexagonal tiling of the plane using hexagons of diameter slightly less than 1 (there is room for error), and 7-coloring them in a fairly obvious way, left to the reader to discover.

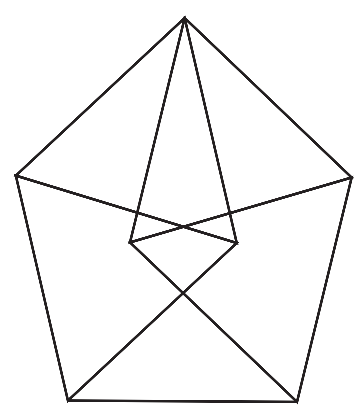

Figure 2: Proof that $4 \leq \chi(\mathbb{E}^2)$.

Given that the known bounds are so easy to prove, it may be surprising that the problem has proven to be so stubborn. In 1981, Falconer showed that if we assume the axiom that all subsets of $\mathbb{E}^n$ are Lebesgue measurable, then $\chi(\mathbb{E}^2) \geq 5$. For other conditional results about $\chi(\mathbb{E}^2)$, see [26]. Other results about the chromatic number of $\mathbb{E}^n$ include the bounds $6 \leq \chi(\mathbb{E}^3) \leq 15$ ([5],[17]) and $7 \leq \chi(\mathbb{E}^4) \leq 49$ ([11]).

In [24], a variant of this problem is discussed (and tentatively attributed to Erdős). Say that a set S in $\mathbb{E}^2$ *realizes distance* d if some two points $x, y \in S$ are distance d apart. The *polychromatic number of the plane* is the least number of colors $\chi_p(\mathbb{E}^2)$ such that it is possible to color the plane with $\chi_p(\mathbb{E}^2)$ colors so that no color realizes all distances. Of course, $\chi_p(\mathbb{E}^2) \leq \chi(\mathbb{E}^2)$, since in the latter case no color realizes distance 1. The bounds $4 \leq \chi_p(\mathbb{E}^2) \leq 6$ are due to Stechkin and published in [21], though as with $\chi(\mathbb{E}^2)$, the determination of the actual number $\chi_p(\mathbb{E}^2)$ is open.

The chromatic number of rational space has also been studied. In [2], M. Benda and M. Perles show that $\chi(\mathbb{Q}^2) = 2$, $\chi(\mathbb{Q}^3) = 2$, and $\chi(\mathbb{Q}^4) = 4$. The authors pose some problems in their conclusion: try to determine $\chi(\mathbb{Q}^n)$ for some $n \geq 5$ (in [4], K. B. Chilakamarri shows that $\chi(\mathbb{Q}^5) \geq 6$), or try to find $\chi(X^2)$ where X is some algebraic extension of $\mathbb{Q}$, perhaps $\mathbb{Q}[\sqrt{2}]$.

The question of which graphs are unit distance graphs in $\mathbb{E}^2$ is also open. For instance, it is easy to see that the graph K_4 cannot be a unit distance graph in $\mathbb{E}^2$, but it is not known if any particular subgraphs are excluded. In [3] a problem is posed: must every bipartite graph that is not a unit distance graph contain $K_{2,3}$ as a subgraph? The answer is no: it is not very difficult to show that the 5-dimensional hypercube, Q_5, with all 16 of its space diagonals attached, is a counterexample. Here is a sketch of the proof:

Sketch of proof. The 2-dimensional hypercube, embedded in the plane as a unit distance graph, clearly has to have some two of its opposite vertices at least distance $\sqrt{2}$ apart (opposite in the sense that they are maximally distant pairs in Q^2). The same is true of Q_5. Now let G be the graph Q_5 with all of its space diagonals added (connect two vertices by an edge if they are distance 5 apart in Q_5). Since some two of the vertices of Q_5 in a unit embedding are necessarily farther than unit distance apart, we cannot embed G as a unit distance graph in $\mathbb{E}^2$. Moreover, G is bipartite and contains no copy of $K_{2,3}$ as a subgraph. □

Paul O'Donnell has shown in [18] and [19] that there exist 4-chromatic unit distance graphs of arbitrary girth. Since a complete characterization of the unit distance graphs in the plane would immediately determine the value of $\chi(\mathbb{E}^2)$, this is probably a very difficult task; still, it would be interesting to know what can be said.

Finally, we ask a basic question about unit distances in the plane: how dense can a Lebesgue measurable set S be in $\mathbb{E}^n$ if it avoids unit distance? A good first attempt in $\mathbb{E}^2$ is to tile the plane with hexagons whose centers are distance 2 apart, and let S be the set of open circles of diameter 1 centered in the hexagons; this achieves density $\frac{\pi}{8\sqrt{3}} > 0.2267$. However, in 1967, Croft showed in [7] that by modifying this coloring slightly it is possible to achieve a density of more than 0.2294. Coulson and Payne examined the same problem in $\mathbb{E}^3$ [6], but there has been no improvement over Croft's result in the case of $\mathbb{E}^2$.

4 More General Distance Graphs

There are also many open problems about graphs more general than unit distance graphs – here we will only consider Euclidean n-space. For $A \subseteq \mathbb{R}$, let $G^A(\mathbb{E}^n)$ be the graph in $\mathbb{E}^n$ with vertex set $\mathbb{E}^n$ and edge set $\{x, y \in \mathbb{E}^n : \|x - y\| \in A\}$. Very recently Ardal, Maňuch, Rosenfeld, Shelah and Stacho have shown in [1] that if we let X be the set of all odd integers, then $\chi(G^X(\mathbb{E}^2)) \geq 5$, but the current upper bound on this number is the trivial bound $\aleph_0$.

In related work, L. Ivanov has considered the chromatic number of $G^{[1,d]}(\mathbb{E}^n)$ for $d > 1$ [12]. For this generalization of the unit distance graph, very little is currently known. Similar questions are pursued in [20], in which the authors present new results about $\chi(G^A(\mathbb{R}^n))$ and $\chi(G^A(\mathbb{Q}^n))$ for $|A| \in \{2, 3, 4\}$.

These problems are interesting partly because so little is known. Kuratowski's theorem [16] beautifully classifies the planar graphs, but there is no such theorem for unit graphs. The unit distance graph in the plane (and we have no need to be more general here) is simple enough to describe to a non-mathematician, and so enigmatic that finding its chromatic number is a new four color map problem for graph theorists.

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