A note on marking lines in $[k]^n$

In honor of Rick Wilson's 65th birthday

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Abstract

In $[k]^n = [k]\times[k]\times\cdots\times[k]$, a coordinate line consists of the collection of points where all but one coordinate is fixed and the unfixed coordinate varies over all possibilities. We consider the problem of marking (or designating) one point on each line in $[k]^n$ so that each point in $[k]^n$ is marked either a or b times, for some fixed a or b. This is equivalent to forming a strategy for a hat guessing game for n players with k different colors of hats where the number of correct guesses, regardless of hats placed, is either a or b.

If we let $s \geq 0$ and $t \geq 0$ denote the number of vertices marked a and b times respectively, then we have the following obvious necessary conditions: $s + t = k^n$ (the number of points) and $as + bt = nk^{n-1}$ (the number of lines). Our main result is to show for $n \leq 5$, and k arbitrary, that these necessary conditions are also sufficient.

1 Introduction

In a hat guessing game there are n players and on each player a hat will be placed on his or her head of one of k colors. Each player can then see all other hats but his or her own and then each player independently and simultaneously must guess the color of the hat they are wearing. There are some other variations on this game, such as players might only see some subset of the other players, or they may also be allowed to pass on their guess, but for our purposes we will stay in this simplified setting.

Before hats are placed on the players the players are allowed to get together and form a strategy. The strategy they form will vary with the goal they are trying to achieve. For instance, they might want to guarantee that at least $\lfloor n/k \rfloor$ of the players guess correctly, or that the correct guesses are proportional to the various hat colors which have been placed. More information about hat guessing games and strategies can be found in [2].

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A way to describe a strategy in the game is by *marking coordinate lines* in $[k]^n$. A *coordinate line* in $[k]^n$ is a set of k points in which $n-1$ of the coordinates are fixed and the remaining coordinate takes on all k possible values (this is a special case of a "combinatorial line" which is allowed multiple unfixed coordinates but they move together). By *marking a line* we mean to designate a point on that line.

In particular, the points in $[k]^n$, i.e., $(x_1, x_2, \ldots, x_k)$, correspond to the k^n different possible placement of hats. Namely, where the first player has the hat of type x_1, the second player has the hat of type x_2, and so on. A strategy must describe how each player will guess in each possible situation that player might be in. The ith player sees all hats but their own, and therefore know that the placement lies on the line $(x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_n)$, where the $* \in \{1, 2, \ldots, k\}$, and the strategy then is to pick the point on the line where the ith entry is the guess of the player, i.e., to mark the line. Each possible line will have exactly one point on that line marked, and the number of times that a point is marked corresponds to the number of correct guesses that will be made when that corresponding placement of hats occurs.

(This type of marking of points corresponds to a deterministic strategy, that is we do not allow for randomness in guessing. However, it is not difficult to allow for randomness. This is done by instead of marking a designated point on the line we give a probability distribution to the points on that line (i.e., assign nonnegative weights with sum one to the points on the line). In this setting the sum of the weights for the lines passing through a point is then the expected number of correct guesses that will be made when that corresponding placement of hats occurs. For our purposes in this note we will be working only with deterministic strategies.)

H. Iwasawa [3] posed the question about forming a strategy so that the number of correct guesses is either a or b, regardless of how the hats are placed on the players. In general, if we let $s \geq 0$ denote the number of placements that will have a correct guesses and $t \geq 0$ denote the number of placements that will have b correct guesses then we must have the following necessary conditions:

$$\begin{aligned} s + t &= k^n && \text{the total number of points in } [k]^n; \\ as + bt &= nk^{n-1} && \text{the total number of lines in } [k]^n. \end{aligned}$$

Theorem 1 (Buhler, Butler, Graham and Tressler [1]). *When there are n players and $k = 2$ colors of hats then there is a strategy so that the number of correct guesses is either a or b (equivalently, there is a way to mark the lines in $[2]^n$ so that each point is marked either a times or b times) if and only if there are nonnegative integers s and t so that $s + t = 2^n$ and $as + bt = n2^{n-1}$.*

In other words, the necessary conditions are also sufficient in the special case when $k = 2$. An open problem mentioned at the end of the Buhler et al. paper [1] is to then decide if the necessary conditions are always sufficient for arbitrary k. In this note we will give some support to this. Our main result is the following.

Theorem 2. *When there are $n \le 5$ players and k colors of hats, k arbitrary, then there is a strategy so that the number of correct guesses is either a or b (equivalently, there is a way to mark the lines in $[k]^n$ so that each point is marked either a times or b times) if and only if there are nonnegative integers s and t so that $s + t = k^n$ and $as + bt = nk^{n-1}$.*

Another alternative way to state Theorem 2 is that if G_k^n is the bipartite graph with vertices the points and lines of $[k]^n$ and edges consist of pairs of points and lines such that the point lies on the line, then there is some subset of edges such that the degree of a vertex corresponding to a line is 1 and the degree of a vertex corresponding to a point is a or b.

Throughout this note we will use the notation $[a, b]_k^n$ as a shorthand for a realization of a marking of the lines $[k]^n$ where each point is marked either a or b times; we will also assume that $a \le b$.

2 Proof of Theorem 2

In this section we will establish Theorem 2. We will first show how to build "larger" markings from a given marking, which will allow us to reduce the number of cases that need to be considered. We then establish it in a few general cases and finally clean up the few remaining cases that will remain.

Inflated markups

We first look at some various ways that markings can be inflated to give markings in larger cases. This will allow us to reduce down to a smaller number of base cases, namely those that cannot be found by inflating some smaller marking.

Proposition 3. *Given $[a, b]_k^n$ then we also have the following:*

(a) $[a + 1, b + 1]_k^{n+k}$;

(b) $[\ell a, \ell b]_k^{\ell n}$ *for all* $\ell \ge 2$.

Given $[0, b]_k^n$ *then we also have the following:*

(c) $[0, b]_{\ell k}^n$ *for all* $\ell \ge 2$.

Proof. First we note that in $[k]^k$, the number of lines equals the number of points, and so it is possible to mark the lines in such a way that each point is marked exactly once. One way to do this is to mark the line $(y_1, \ldots, y_{i-1}, *, y_{i+1}, \ldots, y_k)$ at the point $(y_1, \ldots, y_{i-1}, b, y_{i+1}, \ldots, y_k)$ where b is chosen so that

$$y_1 + \cdots + y_{i-1} + b + y_{i+1} + \cdots + y_k \equiv i \pmod{k}.$$

Clearly, each line marks a unique point, while on the other hand the point $(y_1, \ldots, y_k)$ is marked by the line where the $y_1 + \cdots + y_k \pmod{k}$ entry is allowed to vary, so each point is marked at least once and hence exactly once.

Next we note that $[k]^n \times [k]^k = [k]^{n+k}$ and we can combine the respective markings together, i.e., we mark $(x_1, \ldots x_{j-1}, *, x_{j+1}, \ldots, x_n, y_1, \ldots, y_k)$ as dictated by the line in the smaller $[k]^n$, namely as $(x_1, \ldots x_{j-1}, *, x_{j+1}, \ldots, x_n)$ is marked in $[a, b]_k^n$, and similarly, we mark $(y_1, \ldots, y_n, y_1, \ldots, y_{i-1}, *, y_{i+1}, \ldots, y_k)$ as dictated by the line $(y_1, \ldots, y_{i-1}, *, y_{i+1}, \ldots, y_k)$ in the $[k]^k$ as given above. Each point in $[k]^{n+k}$ will be marked either a or b times from the lines dictated by $[k]^n$ and exactly once from the lines dictated by $[k]^k$, so each point will have been marked either $a+1$ or $b+1$ times giving the desired $[a+1, b+1]_k^{n+k}$.

For the second property, we observe that $[k]^{\ell n} = [k]^\ell \times [k]^\ell \times \cdots \times [k]^\ell$. We will find it convenient to describe points in $[k]^{\ell n}$ by $(w_1, \ldots, w_n)$ where $w_i = (x_1^{(i)}, \ldots, x_\ell^{(i)})$, and let

$$\psi(w_i) = \begin{cases} x_1^{(i)} + \cdots + x_\ell^{(i)} & \text{if } w_i \text{ does not contain } *; \text{ and} \\ *' & \text{if } w_i \text{ does contain } *. \end{cases}$$

We can use this to define a map $\Psi : [k]^{\ell n} \mapsto [k]^n$ by

$$\Psi\big((w_1, \ldots, w_n)\big) = \big(\psi(w_1), \ldots, \psi(w_n)\big).$$

This map can be used to determine how we mark lines in $[k]^{\ell n}$. For a line in $[k]^{\ell n}$ given by $(w_1, \ldots, w_n)$ and where $x_j^{(i)} = *$, we apply Ψ to find the corresponding line in $[k]^n$, namely,

$$\big(\psi(w_1), \ldots, \psi(w_{i-1}), *', \psi(w_{i+1}), \ldots, \psi(w_n)\big).$$

Using the marking $[a, b]_k^n$ we now determine how to mark this line, suppose that we mark it so that $*' = c$, then, we mark our original line by choosing $x_j^{(i)}$ so that

$$x_1^{(i)} + \cdots + x_\ell^{(i)} \equiv c \pmod{k}.$$

With this convention it is easy to see that for every time the point $\big(\psi(w_1), \ldots, \psi(w_n)\big)$ is marked in $[k]^n$, the point $(w_1, \ldots, w_n)$ is marked ℓ times in $[k]^{\ell n}$. Namely, if the point is marked by a line where it is the ith entry that varies in $[k]^n$ then in $[k]^{\ell n}$ it is marked by the lines where we vary each of the ℓ entries in w_i. This gives us the desired $[\ell a, \ell b]_k^{\ell n}$.

For the third property, we observe that we have $[0, n]_\ell^n$ for any ℓ. To see this it suffices to find ℓ^{n-1} points no two of which are on the same line. These points are then marked on all lines that pass through them, as a result we will have marked all lines and have the $[0, n]_\ell^n$. One way to do this is as

$$(x_1, x_2, \ldots, x_{n-1}, x_1 + \cdots + x_{n-1} \pmod{\ell}),$$

i.e., where $x_1, \ldots, x_{n-1}$ are arbitrary and the last entry acts as parity term. To check that no two of these points are on the same line we only need to make sure that they differ in at least two coordinates. If they differ in at least two of the $x_1, \ldots, x_{n-1}$ then we are done. Otherwise, they differ in one of $x_1, \ldots, x_{n-1}$ and so they must also differ in the last coordinate (the parity) entry and so differ in two entries.

Now starting with $[\ell k]^n$ we can subdivide it into ℓ^n smaller subcubes of side length k and using the $[0,n]^n_\ell$ we can select out ℓ^{n-1} of these subcubes so that each line in $[\ell k]^n$ passes through exactly one of these subcubes. Finally, using the marking $[0,b]^n_k$ we mark each one of these selected subcubes. Every line will be used exactly once and each point inside of the selected subcubes will be marked either 0 or b times while each point outside of the selected subcubes will be marked 0 times. This gives our desired $[0,b]^n_{\ell k}$. □

Some markings

We now turn to the problem of finding some simple markings.

Proposition 4. *If* $(b-a)\,\big|\,(kb-n)$ *and* $a \le n/k \le b$ *then we have* $[a,b]^n_k$.

Proof. Taking our necessary conditions in the introduction and solving for s and t we have

$$s = \left(\frac{kb-n}{b-a}\right)k^{n-1} \qquad \text{and} \qquad t = \left(\frac{n-ka}{b-a}\right)k^{n-1}.$$

By our assumptions we have that $s = pk^{n-1}$ and $t = qk^{n-1}$ for nonnegative integers p and q, where the necessary conditions give $p+q=k$ and $ap+bq=n$.

Instead of producing a rule for marking lines, we can equivalently give a rule for which line(s) mark a given point (being careful to ensure that each line is used exactly once). One way to indicate which line is marking a given point is by indicating which is the non-fixed coordinate of the line which marks the points. To do this we will first group the points into k sets P_i for $1 \le i \le k$, where P_i contains all points of the form

$$(x_1, x_2, \ldots, x_{n-1}, x_n) \qquad \text{with } x_1 + \cdots + x_n \equiv i \pmod{k}.$$

Each P_i has exactly k^{n-1} points and further among the points in P_i no two lie on a line (using the same argument as in Proposition 3).

Now for P_i with $1 \le i \le p$ we will let the points in P_i be marked by the lines which vary in the entries $1+(i-1)a \le j \le ia$; while for P_i with $p+1 \le i \le p+q$ we will let the points in P_i be marked by the lines which vary in the entries $pa+1+(i-p-1)b \le j \le pa+(i-p)b$.

Clearly, each point has been marked either a or b times. It only remains to check that we have used each line exactly once. Since the total of markings equals the total number of lines we can do this by checking that no line was used more than once. Clearly we cannot have used the same line twice in a single P_i since no two points in P_i lie on the same line. Similarly we cannot have used the same line in P_i as in P_j, with $j \ne i$, since they must vary in different entries. Therefore, no line could have been used twice. □

As a special case we have the following.

Corollary 5. *If* $n \le k$ *then we have* $[0,1]^n_k$.

More generally, we would like to have $[0,b]^n_k$ whenever the necessary conditions are satisfied. This is more difficult since Proposition 4 will usually not apply. We do however have the following case.

Proposition 6. *If $1 < n \le 2k$ and nk is even then we have $[0,2]^n_k$.*

Proof. By our assumptions, we will satisfy the necessary conditions. If n is even, then we satisfy Proposition 4 and we are done. So we may now assume that n is odd, i.e., $n = 2N+1$, for some N. But we still need that $s = (4-n)k^{(n-1)}/2$ is even, so we can conclude that k is even, i.e., $k = 2K$ for some K. Further, we have $k \ge n/2 = N + 1/2$ and so $k \ge N+1$.

Let $F_0 = \{0,\dots,K-1\}$ and $F_1 = \{K,\dots,2K-1\}$ (i.e., we partition $0,\dots,k-1$ into a first and second half). We now collect the points into n groups. Further, for each group of points we indicate which lines will mark these points by boxing the corresponding set of entries that would vary to produce the line. In the table f_0 represents an element of F_0, f_1 represents an element of F_1 and x_i is arbitrary.

$$\begin{array}{cccccccccl}
\boxed{f_0} & \boxed{x_1} & x_2 & x_3 & x_4 & x_5 & \cdots & x_{n-2} & x_{n-1} & \text{with } f_0 + \sum x_i \equiv 0 \pmod{k} \\
f_1 & \boxed{x_1} & \boxed{x_2} & x_3 & x_4 & x_5 & \cdots & x_{n-2} & x_{n-1} & \text{with } f_1 + \sum x_i \equiv 0 \pmod{k} \\
f_0 & x_1 & \boxed{x_2} & \boxed{x_3} & x_4 & x_5 & \cdots & x_{n-2} & x_{n-1} & \text{with } f_0 + \sum x_i \equiv p_1 \pmod{k} \\
f_1 & x_1 & x_2 & \boxed{x_3} & \boxed{x_4} & x_5 & \cdots & x_{n-2} & x_{n-1} & \text{with } f_1 + \sum x_i \equiv p_1 \pmod{k} \\
f_0 & x_1 & x_2 & x_3 & \boxed{x_4} & \boxed{x_5} & \cdots & x_{n-2} & x_{n-1} & \text{with } f_0 + \sum x_i \equiv p_2 \pmod{k} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & & \\
f_1 & x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & \boxed{x_{n-2}} & \boxed{x_{n-1}} & \text{with } f_1 + \sum x_i \equiv p_{N-1} \pmod{k} \\
\boxed{f_0} & x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_{n-2} & \boxed{x_{n-1}} & \text{with } f_0 + \sum x_i \equiv K \pmod{k}
\end{array}$$

Where we choose $p_1,\dots,p_{N-1}$ so that $0,p_1,\dots,p_{N-1},K$ are all distinct modular k (since $k \ge N+1$ this is possible). In particular, no point will appear on more than one line since it cannot differ in the first entry or satisfy two modular conditions simultaneously.

In each line the modular condition can be viewed as putting a parity entry for x_{n-1} and so for the first entry we have $K = k/2$ choices, and then for the next $n-2$ entries we have k choices for each entry giving us a collection of $(1/2)k^{k-1}$ points on each line. Since there are n lines (which can be counted by the location of the first entry which can vary) we have a total of $(n/2)k^{k-1}$ points in our above list, and each point is marked by two lines.

It now suffices to show that no line marks more than one point to finish showing that this gives our desired $[0,2]^n_k$. We note that any for any set of points for which the sum of the entries is fixed, no two of them can lie on a single line (this takes care of pairs of consecutive lines starting at the top). Further, if the first entries do not match and it is not the first entry which varies then these cannot be the same line (this takes care of pair of consecutive lines starting one line down from the top).

Since we must vary in the same entry the only possibility that remains to check is that the same line occurs in the first and last group as given above. In order for this to happen we would need to have two points

$$\begin{array}{ll}
(f_0, x_1, \dots, x_{n-1}) & \text{with } \; f_0 + \sum x_i \equiv 0 \pmod{k}, \\
(f_0', x_1, \dots, x_{n-1}) & \text{with } \; f_0' + \sum x_i \equiv K \pmod{k},
\end{array}$$

and where all of the x_i match. Taking the difference this would imply

$$f_0' - f_0 \equiv K \pmod{k = 2K}.$$

But $f_0, f_0' \in F_0 = \{0, 1, \dots, K-1\}$, and the elements that can occur in $F_0 - F_0 \pmod{k}$ are $\{0, 1, \dots, K-1, K+1, \dots, 2K-1\}$, in particular this does not contain K. So this cannot happen. □

Finishing off the remaining cases for $n \leq 5$

We still have several cases left. In Table 1 we list markings for $n \leq 10$ which do not already follow from what we have done above or which would follow from a smaller case (i.e., using Proposition 3).

n	Remaining cases
2	
3	
4	$[0,3]_3^4$, $[1,4]_3^4$
5	$[0,3]_3^5$, $[1,4]_3^5$, $[1,3]_4^5$, $[1,5]_4^5$
6	$[1,5]_4^6$, $[0,5]_5^6$, $[1,6]_5^6$
7	$[0,3]_3^7$, $[1,7]_3^7$, $[1,5]_4^7$, $[1,7]_4^7$, $[0,5]_5^7$, $[1,6]_5^7$, $[0,6]_6^7$, $[1,3]_6^7$, $[1,4]_6^7$, $[1,5]_6^7$, $[1,7]_6^7$
8	$[0,3]_3^8$, $[0,5]_5^8$, $[1,6]_5^8$, $[1,4]_6^8$, $[1,5]_6^8$, $[1,7]_6^8$, $[0,7]_7^8$, $[1,8]_7^8$
9	$[1,9]_4^9$, $[0,5]_5^9$, $[1,6]_5^9$, $[1,5]_6^9$, $[1,7]_6^9$, $[1,9]_6^9$, $[0,7]_7^9$, $[1,8]_7^9$, $[1,3]_8^9$, $[1,5]_8^9$, $[1,9]_8^9$, $[0,4]_{2p}^9$ for p prime
10	$[0,9]_3^{10}$, $[1,10]_3^{10}$, $[1,9]_4^{10}$, $[1,7]_6^{10}$, $[1,9]_6^{10}$, $[1,10]_6^{10}$, $[0,7]_7^{10}$, $[1,8]_7^{10}$, $[1,5]_8^{10}$, $[1,9]_8^{10}$, $[1,9]_8^{10}$, $[0,3]_9^{10}$, $[1,4]_9^{10}$, $[1,10]_9^{10}$, $[0,3]_{3p}^{10}$ for p prime

Table 1: The remaining cases for marking lines for $n \leq 10$.

(In generating this table we have taken advantage of the fact that if $k \leq n$ there are only finitely many cases. For $k > n$, the number of points is more than the number of lines and therefore it follows that $a = 0$, i.e., we are in the case $[0, b]_k^n$. By Proposition 3(b) if b and n are not relatively prime then we can deduce it from a simpler case. Also, by Proposition 3(c) if we have $[0, b]_b^n$ then we are done. What remains is then easily worked out.)

To finish the remaining cases for small n we will make use of the following proposition.

Proposition 7. *If there is a filling of the cells of a $k \times k$ array with $1, 2, \dots, n-2, \sigma_1, \sigma_2$ such that:*

- *each cell has either a or b distinct elements;*
- σ_1 *occurs exactly once in each column;*
- σ_2 *occurs exactly once in each row;*
- i *occurs exactly once in each diagonal (allowing wraparound) for* $1 \le i \le n-2$*;*

then we can produce a marking for $[a,b]_k^n$.

To see why this is true, recall that in Proposition 4 we first grouped the points into several large sets by using a parity term, and then described which lines marked each set of points. We will now use something similar, but instead of using one parity term we will use two parity terms. In general, we will have k^2 groups of points which can be expressed as

$$(x_1,\ldots,x_{n-2},\underbrace{x_1+\cdots+x_{n-2}+i}_{=\Sigma_i},\underbrace{x_1+\cdots+x_{n-2}+j}_{=\Sigma_j})=(x_1,\ldots,x_{n-2},\Sigma_i,\Sigma_j).$$

We will find this convenient to place these groups into a $k\times k$ array as

$(x_1,\ldots,x_{n-2},\Sigma_1,\Sigma_1)$	$(x_1,\ldots,x_{n-2},\Sigma_1,\Sigma_2)$	$\cdots$	$(x_1,\ldots,x_{n-2},\Sigma_1,\Sigma_k)$
$(x_1,\ldots,x_{n-2},\Sigma_2,\Sigma_1)$	$(x_1,\ldots,x_{n-2},\Sigma_2,\Sigma_2)$	$\cdots$	$(x_1,\ldots,x_{n-2},\Sigma_2,\Sigma_k)$
$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$
$(x_1,\ldots,x_{n-2},\Sigma_k,\Sigma_1)$	$(x_1,\ldots,x_{n-2},\Sigma_k,\Sigma_2)$	$\cdots$	$(x_1,\ldots,x_{n-2},\Sigma_k,\Sigma_k)$

.

Finally, we need to describe which lines will mark each set of points. To do this we again will indicate the entry(s) which vary to make the line which will mark the point. We will indicate this with $1,2,\ldots,n-2,\sigma_1,\sigma_2$ where i indicates we vary the ith entry, σ_1 indicates we vary the first parity term and σ_2 indicates we vary the second parity term. For example, if we want to find a marking for $[0,3]_3^4$ we can mark the entries as follows:

$1,\sigma_1,\sigma_2$		
	$2,\sigma_1,\sigma_2$	$1,2,\sigma_1$
	$1,2,\sigma_2$	

.

So for example the set of points of the form $(x_1,x_2,\Sigma_3,\Sigma_2)$ will be marked by lines which vary in the entries corresponding to x_1, x_2 and Σ_2, while the set of points of the form $(x_1,x_2,\Sigma_2,\Sigma_1)$ will not be marked by any lines.

Clearly, if we use the marking as indicated above we will mark every point either 0 or 3 times. What we need to make sure of now is that every line was marked once and only once. One way to do this is to make sure we have the correct number of lines present and that no line was used twice. To the first issue, the number of lines which have some fixed entry which varies is k^{n-1}. Since there are k^{n-2} points in each group then we need to have each entry show up k times as being used to mark a set of vertices.

To make sure that no line is marked twice we consider how the same line could occur in markings of two different set of points and then exclude all of these possibilities. Because of the parity terms we can, as in Proposition 4, rule out the possibility of the same line being marked twice in a given set of points. So we need to worry about marking the same line using two different sets of points. If it is the same line being marked then we must have that the same entry is varying and so we have three general cases.

- We vary the first parity term, i.e., $(x_1, \ldots, x_{n-2}, *, \Sigma_j)$. This can occur if we let the first parity term be arbitrary. Therefore, we must avoid having σ_1 occur more than once in a given *column*.
- We vary the second parity term, i.e., $(x_1, \ldots, x_{n-2}, \Sigma_j, *)$. This can occur if we let the second parity term be arbitrary. Therefore, we must avoid having σ_2 occur more than once in a given *row*.
- We vary one of the other entries, i.e., $(x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_{n-2}, \Sigma_i, \Sigma_j)$. This can occur if the parity terms are both shifted by the same amount. For example

  $$(x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_{n-2}, \Sigma_i, \Sigma_j) = (x_1, \ldots, x_{i-1}, * - \ell, x_{i+1}, \ldots, x_{n-2}, \Sigma_{i+\ell}, \Sigma_{j+\ell}).$$

  Therefore, we must avoid having i occur more than once in a given *diagonal* (where a diagonal wraps around the array).

It is now easy to check that the example we gave above will produce a marking for $[0,3]^4_3$. In Table 2 we produce some arrays satisfying Proposition 7 which cover the remaining cases for $n \le 5$, finishing the proof of Theorem 2. Additional examples are given in the appendix.

$1,2,\sigma_1,\sigma_2$	1	1
σ_2	σ_1	2
σ_2	2	σ_1

(a) $[1,4]^4_3$

$1,\sigma_1,\sigma_2$	$1,2,3$	$1,2,3$
	$2,\sigma_1,\sigma_2$	
		$3,\sigma_1,\sigma_2$

(b) $[0,3]^5_3$

$1,2,3,\sigma_1$	$1,2,3,\sigma_2$	1
2	σ_1	σ_2
σ_2	3	σ_1

(c) $[1,4]^5_3$

$1,2,3$	$1,2,3$	σ_2	1
σ_1	σ_1	σ_2	1
2	2	σ_1	σ_2
σ_2	3	3	σ_1

(d) $[1,3]^5_4$

$1,2,3,\sigma_1,\sigma_2$	1	1	1
σ_2	σ_1	2	2
σ_2	2	σ_1	3
σ_2	3	3	σ_1

(e) $[1,5]^5_4$

Table 2: Arrays satisfying Proposition 7.

We can also use Proposition 7 to establish some general cases. For example, the following array shows we always have a marking for $[1,n]^n_{n-1}$:

$1,\dots,n-2$	1	1	1	1	$\cdots$	1
σ_2	σ_1	2	2	2	$\cdots$	2
σ_2	2	σ_1	3	3	$\cdots$	3
σ_2	3	3	σ_1	4	$\cdots$	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$
σ_2	$n-2$	$n-2$	$n-2$	$n-2$	$\cdots$	σ_1

Similarly, the following array shows we always have a marking for $[1,n-1]^n_{n-2}$:

$1,\dots,n-2,\sigma_1$	$1,\dots,n-2,\sigma_2$	1	1	1	$\cdots$	1
2	σ_1	σ_2	2	2	$\cdots$	2
3	3	σ_1	σ_2	3	$\cdots$	3
4	4	4	σ_1	σ_2	$\cdots$	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$
σ_2	$n-2$	$n-2$	$n-2$	$n-2$	$\cdots$	σ_1

3 Conclusion

We have shown that for $n \le 5$ that it is possible to mark the lines in $[k]^n$ so that each point in $[k]^n$ is marked either a or b times whenever the obvious necessary conditions hold. Combining this with the known result for $k = 2$ in Buhler et al. [1] gives support for the following.

Conjecture 1. *There is a marking of the lines in $[k]^n$ so that each line marks exactly one point and each point gets marked either a or b times if and only if there are nonnegative integers s and t so that $s + t = k^n$ and $as + bt = nk^{n-1}$.*

One way to work on this would be to keep showing that this holds for some larger values of n (for instance, the only remaining case to establish the result for $n = 6$ is $[0,5]^6_5$). For small cases it is possible to do this by computer, i.e., to rewrite the problem as a SAT problem and then test if it can be satisfied. However, as k and n increases the size of the SAT problem grows large and becomes prohibitive to determine using that method. Some new methods and ideas will likely be needed to establish the conjecture

Instead of trying to establish the conjecture in general one might consider the problem of showing that it holds for the special case $[0,b]^n_k$. This would show, for example, that the conjecture holds for all $k > n$.

Another variation to consider is instead of marking *lines*, we can mark *planes* or higher dimensional subspaces. So we can define a plane by saying that all but two coordinates are fixed and the two non-fixed coordinates can be arbitrary. We then mark one point on each plane and consider the problem of marking planes so that each point is marked either a or b times.

As an example, consider the following marking of points for planes in $[2]^4$.

plane	point	plane	point	plane	point
$(0,0,*,*)$	$(0,0,0,0)$	$(0,*,*,0)$	$(0,1,1,0)$	$(*,0,*,0)$	$(0,0,1,0)$
$(0,1,*,*)$	$(0,1,0,0)$	$(0,*,*,1)$	$(0,0,1,1)$	$(*,0,*,1)$	$(1,0,1,1)$
$(1,0,*,*)$	$(1,0,0,0)$	$(1,*,*,0)$	$(1,1,1,0)$	$(*,1,*,0)$	$(0,1,0,0)$
$(1,1,*,*)$	$(1,1,0,0)$	$(1,*,*,1)$	$(1,1,0,1)$	$(*,1,*,1)$	$(0,1,1,1)$
$(0,*,0,*)$	$(0,0,0,1)$	$(*,0,0,*)$	$(0,0,0,1)$	$(*,*,0,0)$	$(1,0,0,0)$
$(0,*,1,*)$	$(0,0,1,0)$	$(*,0,1,*)$	$(1,0,1,1)$	$(*,*,0,1)$	$(1,1,0,1)$
$(1,*,0,*)$	$(1,0,0,1)$	$(*,1,0,*)$	$(0,1,0,1)$	$(*,*,1,0)$	$(1,1,1,0)$
$(1,*,1,*)$	$(1,0,1,0)$	$(*,1,1,*)$	$(0,1,1,1)$	$(*,*,1,1)$	$(1,1,1,1)$

A quick check shows that each point has been marked either 1 or 2 times. As with marking lines we have some necessary conditions, namely if $s \geq 0$ and $t \geq 0$ are the number of points marked a or b times then $s + t = k^n$ (the number of points) and $as + bt = \binom{n}{2}k^{n-2}$ (the number of planes).

Problem 2. *Are the necessary conditions for marking planes also sufficient?*

One can even consider marking higher dimensional spans, but any insights made into solving this problem for planes would likely generalize to higher dimensional spans as well.

Acknowledgments

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References

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Appendix: More arrays satisfying Proposition 7

$1,\sigma_1,\sigma_2$	$3,4,5$	$1,2,\sigma_1$
$3,4,5$	$2,3,\sigma_1$	$1,2,\sigma_2$
		$4,5,\sigma_2$

(a) $[0,3]_3^7$

$1,\sigma_1,\sigma_2$	$1,2,3$	$1,2,3$
$4,5,6$	$2,3,\sigma_1$	$4,\sigma_1,\sigma_2$
$5,6,\sigma_2$		$4,5,6$

(b) $[0,3]_3^8$

$1,2,\sigma_1,\sigma_2$		$4,5,6,7$	$1,2,3,\sigma_1$
	$3,4,\sigma_1,\sigma_2$		
	$4,5,6,7$	$5,6,7,\sigma_1$	$1,2,3,\sigma_2$
$4,5,6,7$	$1,2,3,\sigma_2$		

(c) $[0,4]_4^9$

1	1	1	1	1	σ_2
σ_2	2	2	2	2	2
3	σ_2	3	3	3	3
4	4	$1,2,\sigma_2$	4	4	4
5	5	5	$3,4,\sigma_2$	5	5
σ_1	σ_1	σ_1	σ_1	$5,\sigma_1,\sigma_2$	σ_1

(d) $[1,3]_6^7$

1	1	1	1	1	σ_2
σ_2	2	2	2	2	2
3	σ_2	3	3	3	3
4	4	σ_2	4	4	4
5	5	5	$1,2,3,\sigma_2$	5	5
σ_1	σ_1	σ_1	σ_1	$4,5,\sigma_1,\sigma_2$	σ_1

(e) $[1,4]_6^7$

$1,2,3,\sigma_1$	$1,2,3,\sigma_2$	1	1	1	1
2	$4,5,6,\sigma_1$	$4,5,6,\sigma_2$	2	2	2
3	3	σ_1	σ_2	3	3
4	4	4	σ_1	σ_2	4
5	5	5	5	σ_1	σ_2
σ_2	6	6	6	6	σ_1

(f) $[1,4]_6^8$

$1,2,3,\sigma_1,\sigma_2$	$1,2,3,4,5$	1	1	1	1
2	$4,5,6,\sigma_1,\sigma_2$	6	2	2	2
3	3	σ_1	σ_2	3	3
4	4	4	σ_1	σ_2	4
5	5	5	5	σ_1	σ_2
σ_2	6	6	6	6	σ_1

(g) $[1,5]_6^8$

$1,\sigma_1,\sigma_2$	1	1	1	1	1	1	1
σ_2	$2,3,\sigma_1$	2	2	2	2	2	2
σ_2	2	$4,5,\sigma_1$	3	3	3	3	3
σ_2	3	3	$6,7,\sigma_1$	4	4	4	4
σ_2	4	4	4	σ_1	5	5	5
σ_2	5	5	5	5	σ_1	6	6
σ_2	6	6	6	6	6	σ_1	7
σ_2	7	7	7	7	7	7	σ_1

(h) $[1,3]_8^9$ (a perturbation will also give $[1,5]_8^9$)