Juggling Mathematics and Magic

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Abstract

The mystery of magic and the artistry of juggling have surprising links to interesting ideas in mathematics. In this note, I will illustrate several of these connections.

1 Perfect shuffles

Playing cards have been around for many centuries, and indeed, their use in various games of chance was one of the motivating factors in the early development of probability theory. However, they have also been a favorite prop for use by magicians throughout the ages. It turns out that many of these magic effects depend only on certain mathematical properties of the arrangements of the cards.

As an example, suppose we start with a deck of $2n$ cards arranged in a stack labeled from top to bottom as $1, 2, \ldots, 2n$. (The usual deck has size $2n = 52$.) Many skilled magicians are able to perform a *perfect* shuffle on this deck. What this means is that the deck is cut into two equal-sized stacks of n cards each, and these two stacks are then interleaved so that their cards exactly alternate in the combined stack. However, there are two possibilities for this shuffle, namely, the top card of the original stack (in position 1) can remain on top in the combined stack, or it can end up in position 2. The first possibility is called an *out-shuffle* while the second is called an *in-shuffle* (see Figure 1). A natural question to ask is exactly *which* possible permutations of the deck can arise when an arbitrary sequence of in- and out-shuffles is

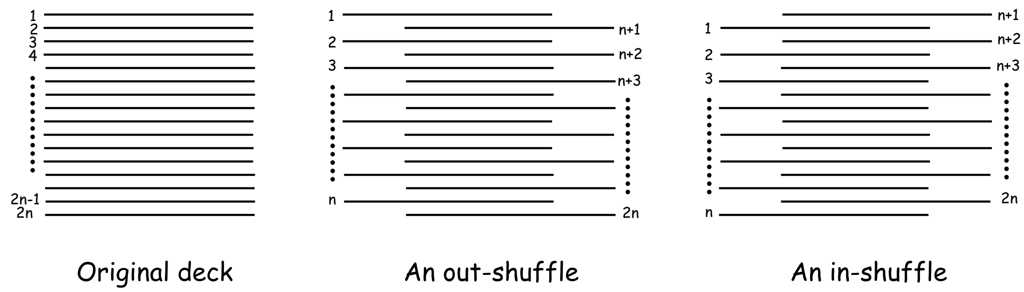

Figure 1: Out- and in-shuffles

performed on an initially ordered deck of size $2n$. A first guess might be that *all possible* $(2n)!$ permutations can be realized by using these shuffles. However, magicians realized long ago that there is natural restriction on the cards when shuffled this way. Namely, *pairs* of cards that are *symmetrically located about the center* will always remain so. In other words, the pair of cards originally in positions $\{i, 2n + 1 - i\}$ will be moved into the pair in positions $\{j, 2n+1-j\}$ for some j in the shuffled deck. However, the card in position i might go to position j or it might go to position $2n+1-j$. Thus, the number of possible arrangements can be at most $2^n n!$ (a factor of 2 for two possible orders of each pair, and a factor of $n!$ for the possible arrangements of the n pairs). Can this many arrangements actually be realized?

From a more mathematical perspective, let $Sh(2n) = \langle I, O \rangle$ denote the group of possible permutations achievable by in- and out-shuffles on a deck of size $2n$. Thus, $Sh(2n)$ is a subgroup of S_{2n}, the symmetric group of all permutations of $\{1, 2, \ldots, 2n\}$. What is the group $Sh(2n)$ and in particular, what is its order $|Sh(2n)|$? We have just pointed out that $|Sh(2n)| \le 2^n n!$

The answer to this question was first given in the paper [2] of Diaconis, Kantor and the author. It turns out that for essentially one-fourth of the possible values of n, $|Sh(2n)| = 2^n n!$. More precisely, this happens exactly for $n \equiv 2 \pmod 4$, $n > 6$. For odd $n > 5$, we lose a factor of 2, and in this case, $|Sh(2n)| = 2^{n-1} n!$. Further, for $n \equiv 0 \pmod 4$, $n > 12$, and n is *not* a power of 2, we lose another factor of 2, and in this case, $|Sh(2n)| = 2^{n-2} n!$. Notice that in all these cases, the number of achievable arrangements is exponentially large as a function of the deck size. This is *not* good news for magicians! However, there is *very good* news when the deck size is $2n = 2^t$ for some t. In this case, the order of $Sh(2^t)$ is just $t \cdot 2^t$ (and for the mathematically inclined, the group $Sh(2n)$ is a semi-direct product $\mathbf{Z}_2^t$ by $\mathbf{Z}_k$; see [2] for details). In particular, the set of possible permutations is so small that knowing the values of any *two* cards (e.g., the top and bottom cards) determines the values of the cards at all the other positions (and you can now understand why magicians often use decks of sizes $16, 32$ or 64).

But you say, what about values not covered by the above ranges? Well, here is another surprise. For a deck of size 24, the group of permutations generated by the in- and out-shuffles on the 12 centrally symmetric pairs is just the Mathieu group M_{12}, the celebrated sporadic simple group of order $8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 = 95040$. So, the magicians of the middle ages could have discovered M_{12} long before mathematicians if they had only been paying attention!

Of course, there are still many unanswered questions concerning the shuffle group $Sh(2n) = \langle I, O \rangle$. For example, what is the *diameter* of $Sh(2n)$? How do you find minimum length sequences of shuffles that connect two given permutations in $Sh(2n)$? What are the corresponding results and questions when three or more equal-sized decks are perfectly shuffled together? These are just a few examples of interesting mathematics lying just below the surface of some mathematical card tricks. For many other similar examples, the reader can consult [3].

2 Juggling Sequences

Mathematics is often described as the science of patterns. Juggling can be thought of the art of controlling patterns in time and space. There has recently been discovered some surprising new connections between juggling and mathematics. Here is one of these connections. Let us imagine time as proceeding in discrete steps labeled $0, 1, 2, \ldots$. At each one of these discrete time points i, we can assign a non-negative integer $t(i)$. In this model, we use a finite sequence $T = (t(i), t(2), \ldots, t(n))$ to specify a sequence of successive throws so that at each time point i, an object (usually called a "ball") is thrown into the air at time i and comes down $t(i)$ units later at time $i + t(i)$. We usually assume that the sequence T is repeated indefinitely. In Fig 2, we show a diagram for the paths of the balls for the sequence $T = (5, 3, 4)$.

Here, this sequence is repeated indefinitely as $(5, 3, 4, 5, 3, 4, 5, 3, 4, \ldots)$.

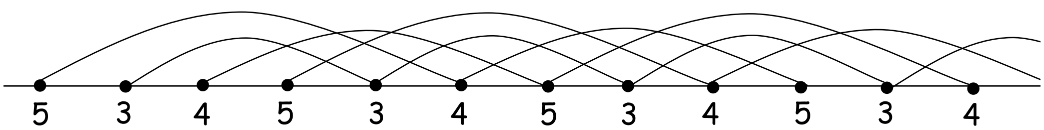

Figure 2: Juggling the sequence $(5, 3, 4, 5, 3, 4, \ldots)$.

The usual interpretation of such patterns is that the balls are thrown an caught alternately with the right and left hands. Thus, if a ball in the pattern is thrown at time i from the *right* hand and it comes down $t(i) = 5$ time steps later at time $i + 5$, then it would land in the *left* hand. Notice that for this pattern, two balls never come down at the same time (a very important consideration for jugglers!). However, as we see in Fig 3, this is not the case for the sequence $(5, 4, 3, 5, 4, 3, \ldots)$.

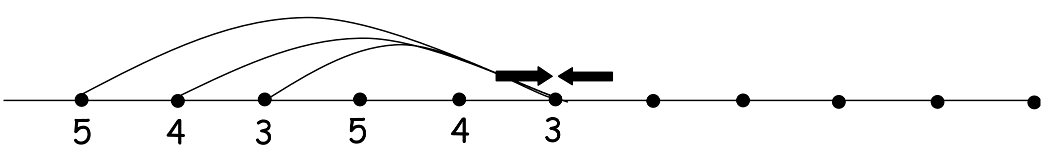

Figure 3: A problem in trying to juggle the sequence $(5, 4, 3, 5, 4, 3, \ldots)$.

It is easy to see that a *necessary* condition for preventing two balls from coming down at the same time is that all of the quantities $i + t(i)$ should be *distinct*. However, this is not sufficient as the sequence $(3, 4, 6)$ shows, for example. Since we repeat the pattern indefinitely, then the sums $i + t(i)$ must actually be distinct modulo n. This turns out to be a necessary and sufficient condition for $(t(1), t(2), \ldots, t(n))$ to be a juggling sequence, and leads to the following definition:

A **Juggling Sequence** is defined to be a finite sequence of non-negative integers $T = (t(1), t(2), \ldots, t(n))$ such that all n of the sums $i + t(i) \pmod n$ are distinct.

Such a juggling sequence is said to have "period" n. An important parameter of a juggling sequence is how many balls it represents (you can imagine why a juggler might want to know this!). It is not too hard to convince oneself that the number of balls b defined by a juggling sequence $(t(1), t(2), \ldots, t(n))$ is just the *average* $\frac{1}{n}\sum_{i=1}^{n} t(i)$. (Question: Why is this an integer?). An interesting question then occurs: *How many* juggling sequences are there with b balls and period n. It was first shown in [1] that this number is exactly $(b+1)^n - b^n$. Thus, for example, the number of different juggling sequences with 3 balls and period 5 is $4^5 - 3^5 = 781$ (which is quite a lot if you are really planning to master them all). No simple proof of this result is known, although from the form of the answer, one suspects that such a proof should exist.

As noted above, for any juggling sequence $(t(1), t(2), \ldots, t(n))$, the sum $\sum_{i=1}^{n} t(i)$ must be divisible by n. This is not a sufficient condition though as the sequence $(5, 4, 3)$, for example, shows. However, there is a beautiful theorem of Marshall Hall [4] which is a type of converse. It states the following.

Theorem 1 *Let* $(a(1), a(2), \ldots, a(n))$ *be an arbitrary sequence of integers such that* $\sum_{i=1}^{n} a(i) \equiv 0 \pmod{n}$. *Then there is some permutation* π *of* $\{1, 2, \ldots, n\}$ *such that all the sums* $i + a(\pi(i)) \pmod{n}$ *are distinct (and therefore form a complete residue system modulo* n*).*

Thus, any sequence of non-negative integers with an *integral* average can be rearranged into a juggling sequence. It would be very nice to have a simple proof of this result (perhaps an inspired reader can find one!).

While the result in Theorem 1 guarantees that some rearrangement of a given sequence with an integral average is a juggling sequence, it doesn't tell you how to find such a rearrangement. For example, the sequence $(1, 7, 19, 14, 1, 8, 20, 8, 6, 3, 11, 12, 18, 9, 14, 8, 11, 10, 19, 22, 7, 8, 17)$ of 23 integers has a sum which is divisible by 23. Thus, by Hall's theorem there is some rearrangement of it that is a juggling sequence. How do we actually find such a rearrangement for this sequence? How many such rearrangements are there? In general, which sequences of length n have the *least* number of rearrangements into juggling sequences? (The sequence of all 0's has the most!).

A very nice conjecture generalizing Theorem 1 (due to Kézdy and Snevily [5]) is the following.

Conjecture. Suppose $1 \leq k < n$ and let $(a(1), a(2), \ldots, a(k))$ be a sequence of k (not necessarily distinct) integers. Then there is some permutation π of $\{1, 2, \ldots, k\}$ such that all the sums $i + a(\pi(i)) \pmod{n}$ are distinct. The case $k = n - 1$ follows from Theorem 1 and it is shown in [5] that the conjecture is also true when $2k \leq n + 1$. In all other cases, it is still unresolved.

3 Concluding Remarks

In this note I have given two examples of the many mathematical challenges that arise in the fields of magic and juggling. Of course, as mathematicians, we tend to see mathematics in everything around us! I believe it is always there. All you have to do is to look for it.

References

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