The digraph drop polynomial

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Abstract

For a weighted directed graph (or digraph, for short), denoted by $D = (V, E, w)$, we define a two-variable polynomial $B_D(x, y)$, called the *drop polynomial* of D, which depends intimately on the cycle structure of the permutations on vertices of D. This polynomial generalizes several other digraph polynomials which have been studied in the literature previously, such as the binomial drop polynomial for posets [2], the binomial drop polynomial for digraphs [7], the path/cycle cover polynomial for digraphs [5] and the matrix cover polynomial [6]. We show that $B_D(x, y)$ satisfies a reduction/contraction recurrence as well as the (somewhat mysterious) reciprocity relation $B_{\bar{D}}(x, y) = (-1)^n B_D(-x - y, y)$, where D has n vertices and $\bar{D} = (V, E, \bar{w})$ is the digraph formed from D by defining $\bar{w}(e) = 1 - w(e)$ for each edge $e \in E$.

1 Introduction

Suppose $P = (V, \prec)$ is a partially-ordered set on a set V of cardinality n. In [2], a polynomial $B_P(x)$ called the *binomial drop polynomial* for P was introduced which was defined as follows. If $\pi : V \to V$ is a permutation on V, we say that π has a *drop* at u if $\pi(u) = v$ and $v \prec u$. Let *drop*(π) denote the number of drops that π has. Then define

$$B_P(x) = \sum_{\pi} \binom{x + drop(\pi)}{n},$$

where π runs over all $n!$ permutations on V. The polynomial $B_P(x)$ turns out to have many connections to a variety of subjects in combinatorics. For example, suppose we define the *incomparability graph* $Inc(P)$ of P to be the graph on V where $\{u, v\}$ is an edge of $Inc(p)$ if and only if u and v are incomparable in P. Let $\chi_{Inc(P)}(x)$ denote the chromatic polynomial of $Inc(P)$ (e.g., see [12]). Then for any positive integer b, $B_P(b) = \chi_{Inc(P)}(b)$. Further, $B_P(-1)$ is equal to the number of permutations π on V which have no drops. This is also equal to the number of acyclic orientations of $Inc(P)$ by a classic theorem of Stanley [12]. In the case that P is a chain, i.e., a linearly-ordered set, then $Inc(P)$ has no edges and we have

$$B_P(x) = x^n = \sum_k \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \tag{1}$$

where $\left\langle {n \atop k} \right\rangle$ denotes the usual Eulerian number. This is usually known as Worpitzky's identity (see [9]).

There is a natural digraph $D = (V, E)$ associated to a poset $P = (V, \prec)$. Namely, the vertex set for D is V. For $u, v \in V$ we have a directed edge $(u, v) \in E$ if and only $v \prec u$ in P. Thus, one might ask if (1) could be extended to more general digraphs. In fact, this can be done as follows for any given digraph D. Suppose $\pi : V \to V$ is a permutation on V. Let us say that π has a *drop* at u if $(u, \pi(u)) \in E$. Denote by $drop(\pi)$ to the the *number of drops* that π has. The following polynomial $B_D(x)$, called the *binomial drop polynomial* for D, which generalizes (1), was studied in [7]:

$$B_D(x) = \sum_\pi \binom{x + drop(\pi)}{n} \tag{2}$$

where $|V| = n$ and π runs over all $n!$ permutations on V. This polynomial can be applied to a more general class of digraphs, called *weighted* digraphs. A weighted digraph $D = (V, E, w)$ is a digraph on the vertex set V with an edge set $E \subseteq V \times V$ in which each edge $e \in E$ is assigned a real-valued *weight* $w(e)$ (possibly 0). In this case, if π is a permutation on V, we generalize the definition of a *drop*(π) by defining

$$drop(\pi) = \sum_{\substack{u \in V \\ e=(u,\pi(u))}} w(e). \tag{3}$$

We can then extend the definition of $B_D(x)$ to weighted digraphs as follows:

$$B_D(x) = \sum_{\pi} \binom{x + drop(\pi)}{n} \tag{4}$$

where, as before, the vertex set V of D has size n. Here, we interpret the binomial coefficient in the sum as

$$\binom{x+\alpha}{n} = n!(x+\alpha)^{\underline{n}} = n!(x+\alpha)(x+\alpha-1)\ldots(x+\alpha+n-1)$$

where we use the *falling factorial* notation $z^{\underline{n}} = z(z-1)(z-2)\ldots(z-n+1)$. It was shown in [7] that $B_D(x)$ has a number of interesting properties, such as a deletion/contraction recurrence similar in form to the well-known deletion/contraction recurrence for the Tutte polynomial (see [14]).

The polynomial we will consider in this paper is a generalization of $B_D(x)$ to a 2-variable polynomial $B_D(x, y)$, called the *drop polynomial* for D.

However, before defining $B_D(x, y)$, we first mention a related digraph polynomial. This is the so-called (path/cycle) *cover* polynomial $C_D(x, y)$ for D (cf. [5]). For a *simple* digraph $D = (V, E)$, that is, one in which all edge weights are 1, we normally think of an edge $e = (u, v) \in E$ as represented by a directed arc going from u to v. A *path/cycle cover* $\mathbf{C}$ of E is a collection of vertex-disjoint paths and cycles which cover all the vertices in V, where we consider a single vertex to be a path of length zero, and a loop (u, u) to be a cycle of length one. Let $c_D(i, j)$ denote the number of path/cycle covers of E consisting of i paths and j cycles. The (path/cycle) *cover polynomial* for D is defined by

$$C_D(x, y) = \sum_{i,j} c_D(i, j) x^{\underline{i}} y^j. \tag{5}$$

The cover polynomial $C_D(x,y)$ was generalized in [6] to weighted digraphs $D=(V,E,w)$ as follows. For a path/cycle cover $\mathbf{C}$ of D, let $p(\mathbf{C})$ denote the *number of paths* in $\mathbf{C}$, and let $cyc(\mathbf{C})$ denote the *number of cycles* in $\mathbf{C}$. By the *weight* $w(\mathbf{C})$ of $\mathbf{C}$ we mean the *product* of all the weights of the edges in $\mathbf{C}$. Then the generalization of $C_D(x,y)$ to weighted digraphs is:

$$C_D(x,y)=\sum_{\substack{\mathbf{C}\\ \text{path/cycle cover}}} x^{\underline{p(\mathbf{C})}}y^{cyc(\mathbf{C})}w(\mathbf{C}). \tag{6}$$

It was shown in [6] that in addition to a deletion/contraction recurrence, $C_D(x,y)$ satisfies a number of other interesting properties such as:

(i) If $D_1=(V_1,E_1,w_1)$ and $D_2=(V_2,E_2,w_2)$ are vertex disjoint digraphs and the digraph D is formed from D_1 and D_2 by joining every $v_1\in V_1$ to every $v_2\in V_2$ by an edge (v_1,v_2) of weight 1, then $C_D(x,y)=C_{D_1}(x,y)C_{D_2}(x,y)$;

(ii) If $\bar{D}=(V,E,\bar{w})$ denotes the weighted digraph formed from $D=(V,E,w)$ by subtracting the weight of each edge $e\in E$ from 1, i.e., $\bar{w}(e)=1-w(e)$, and $|V|=n$, then we have the surprising reciprocity formula:

$$C_{\bar{D}}(x,y)=(-1)^nC_D(-x-y,y). \tag{7}$$

This paper is organized as follows: In Section 2, we will give the definition of drop polynomial $B_D(x,y)$ for a weighted digraph $D=(V,E,w)$ in which each edge $e\in E$ is assigned some *weight* $w(e)$. We will ordinarily assume that $w(e)$ is a real number (possibly 0) although all of our results are valid if $w(e)$ just lies in some commutative ring with identity. Basically, $B_D(x,y)$ is a sum over all permutations π on V where each term of the sum depends on $drop(\pi)$ and the cycle structure of π restricted to D. In Section 3, we discuss a number of useful facts about $B_D(x,y)$. An alternative definition of $B_D(x,y)$ is given in Section 4. The drop polynomials $B_D(x,y)$ are similar to but different from the path-cycle polynomials $C_D(x,y)$. We will show in Section 6 that they coincide if D is a simple digraph. However, for general digraphs, they satisfy different deletion/contraction rules as seen in Section 5. Nevertheless, for both polynomials the same reciprocity theorems hold as shown in Section 7. The drop polynomials have direct connections with permutations on the vertex set and in turn lead to intriguing questions concerning enumeration problems for permutations and partial permutations, some of which will be mentioned in Section 8.

2 The drop polynomial $B_D(x,y)$

As usual, we start with a weighted digraph $D = (V, E, w)$ on a set V of size n (possibly having some edges of weight 0.) For a permutation $\pi : V \to V$ we define $E(\pi)$ to be the set of edges $e = (u, v)$ such that $\pi(u) = v$. Thus, $E(\pi)$ consists of paths and cycles. In this case, *loops*, i.e., edges of the form (u, u), are still considered to be cycles (of length 1). Let $P(\pi)$ denoted the set of edges in paths in $E(\pi)$ and let $Cyc(\pi)$ denote the set of cycles in $E(\pi)$. Also, let $cyc(\pi)$ denote $|Cyc(\pi)|$. More generally, for any subset $S \subseteq E$, we can define $cyc(S)$ in the obvious way, i.e., as the number of cycles formed by the edges in S. Further, we define the *weight* $w(S)$ by:

$$w(S) = \sum_{e \in S} w(e). \tag{8}$$

We finally come to the definition of the drop polynomial $B_D(x,y)$:

$$B_D(x,y) = \sum_{\pi} \sum_{P(\pi) \subseteq S \subseteq E(\pi)} (-1)^{|E(\pi) \setminus S|} \binom{x + w(S)}{n} (y-1)^{cyc(\pi) - cyc(S)} y^{cyc(S)} \tag{9}$$

where π ranges over all permutation of the vertex set V of D and $|V| = n$.

For example, suppose that $D = I(n)$, the digraph on n vertices with *no* edges. Then it is easy to see that in this case, $B_{I(n)}(x,y) = x^{\underline{n}}$, where we recall the falling factorial notation $z^{\underline{n}} = z(z-1)(z-2)\dots(z-n+1)$.

Note that if we substitute $y = 1$ in (9), then $B_D(x,y)$ reduces to the *binomial drop* polynomial $B_D(x)$ in (4). For in this case, the only term in the inner sum in (9) which doesn't vanish is for the choice $S = E(\pi)$, and in this case,

$$\begin{aligned} B_D(x,1) &= \sum_{\pi} \binom{x + w(S)}{n} \\ &= \sum_{\pi} \binom{x + \mathrm{drop}(\pi)}{n} \\ &= B_D(x) \end{aligned}$$

which is (4), as desired.

3 Adding weight 0 edges.

For the general weighted digraphs $D = (V, E, w)$ we have been considering up to now, E is some subset of $V \times V$ with various weights assigned to the edges in E. However, there will be some advantages in being able to assume that E is *all* of $V \times V$, where pairs (u, v) which were not originally in E are replaced by edges $e = (u, v)$ with weight $w(e) = 0$. Of course, the downside of doing this is that now the sums involved in evaluating (9) may have many more terms than before. One upside is that equivalent forms of (9) become simpler (cf. Section 4).

We next show that this modification of D (replacing "non-edges" by edges of weight 0) does not change the value of $B_D(x, y)$. We first show that $B_D(x, y)$ is unchanged if a single pair $(u, v) \notin E$ is replaced by an edge $e = (u, v)$ with $w(e) = 0$. Equivalently, we show that removing an edge of weight 0 does not change $B_D(x, y)$.

Theorem 1. *Suppose $D = (V, E, w)$ is a weighted digraph, and $e_0 \in E$ with $w(e_0) = 0$. Form the digraph $D' = (V, E', w)$ by deleting the edge e_0, i.e., $E' = E \setminus e_0$. Then*

$$B_{D'}(x, y) = B_D(x, y).$$

Proof. When referring to D', we will use $E'(\pi), cyc'(\pi)$, etc., to denote the corresponding parameters for D'. Thus, $|E'| = |E| - 1$. From the definition in (9), we have

$$B_{D'}(x, y) = \sum_{\pi} \sum_{P'(\pi) \subseteq S' \subseteq E'(\pi)} (-1)^{|E'(\pi) \setminus S'|} \binom{x + w(S')}{n} (y-1)^{cyc'(\pi) - cyc'(S')} y^{cyc'(S')}$$

The overall plan is to show that $B_{D'}(x, y)$ reduces to $B_D(x, y)$ when $w(e_0) = 0$. To do this, we are going to expand the terms of $B_{D'}(x, y)$ for each π and show how in each case they either cancel each other out, or correspond to unique terms in $B_D(x, y)$. So consider some fixed permutation π on V.

Case (i). $e_0 \notin E(\pi)$. In this case, for each choice of S' in the sum for $B_{D'}$ we can choose $S = S'$ in the corresponding term in the sum for B_D. Since in this case, $|E'(\pi)| = |E(\pi)|, c'(\pi) = c(\pi), c'(S') = c(S)$, etc., the corresponding terms in $B_{D'}$ and B_D are equal.

Case (ii). $e_0 \in P(\pi)$. In this case, we will always have $e_0 \in S$, since $P(\pi) \subseteq S$. Hence, we can write $S = S' \cup e_0$ for some S' with $P'(\pi) \subseteq S' \subseteq E'(\pi)$. Now we have $|E'(\pi)| = |E(\pi)| - 1, |S'| = |S| - 1, |E'(\pi) \setminus S'| = |E(\pi) \setminus S|, cyc'(\pi) = cyc(\pi)$ and $cyc'(S') = cyc(S)$. Since $w(S) = w(S') + w(e_0)$, then when $w(e_0)$ is 0, the corresponding terms in $B_{D'}$ and B_D are equal in this case as well.

Case (iii). $e_0 \in C(\pi)$. Thus, e_0 belongs to some *cycle* $C = \{e_0, e_1, e_2, \ldots, e_r\}$ in D (where $r = 0$ is allowed). Let C^- denote the set $C \setminus \{e_0\}$. There are now two possibilities for S:

(a) $C \nsubseteq S$. In this case, we will find a unique "mate" S'' for S as follows. If $e_0 \notin S$ then set $S'' = S \cup e_0$; if $e_0 \in S$ then set $S'' = S \setminus e_0$. We will examine the sum of the two terms

$$\sum_{P(\pi) \subseteq S \subseteq E(\pi)} (-1)^{|E(\pi) \setminus S|} \binom{x + w(S)}{n} (y-1)^{cyc(\pi) - cyc(S)} y^{cyc(S)} \qquad (10)$$

and

$$\sum_{E(\pi)\subseteq S''\subseteq E(\pi)} (-1)^{|E(\pi)\setminus S''|}\binom{x+w(S'')}{n}(y-1)^{cyc(\pi)-cyc(S'')}y^{cyc(S'')} \tag{11}$$

in B_D. Suppose $S''=S\setminus e_0$ (the other case is symmetric). Then $|S|=|S''|+1, cyc(S'')=cyc(S)$ and $w(S)=w(S'')+w(e_0)=w(S'')$. Thus, $(-1)^{|E(\pi)\setminus S''|}=-(-1)^{|E(\pi)\setminus S|}$ so that the sum of the two terms (10) and (11) is zero.

(b) $C\subseteq S$. In this case we can write $S=C\cup T$ where $T\subseteq E(\pi)\setminus C$. We have $P'(\pi)=P(\pi)\cup C^-$ and we set $S''=C^-\cup T$. Now we have $cyc(S)=cyc(S'')+1$. Thus, since $|S|=|S''|+1$ then adding (10) and (11), we get

$$\begin{aligned}
&\sum_{P(\pi)\subseteq S\subseteq E(\pi))} (-1)^{|E(\pi)\setminus S|}\binom{x+w(S)}{n}(y-1)^{cyc(\pi)-cyc(S)}y^{cyc(S)}\\
&+\sum_{P(\pi)\subseteq S''\subseteq E(\pi)} (-1)^{|E(\pi)\setminus S''|}\binom{x+w(S'')}{n}(y-1)^{cyc(\pi)-cyc(S'')}y^{cyc(S'')}\\
&=\sum_{P(\pi)\subseteq S''\subseteq E(\pi)} -(-1)^{|E(\pi)\setminus S''|}\binom{x+w(S'')+w(e_0)}{n}(y-1)^{cyc(\pi)-cyc(S'')-1}y^{cyc(S'')+1}\\
&+\sum_{P(\pi)\subseteq S''\subseteq E(\pi)} (-1)^{|E(\pi)\setminus S''|}\binom{x+w(S'')}{n}(y-1)^{cyc(\pi)-cyc(S'')}y^{cyc(S'')}\\
&=\sum_{P(\pi)\subseteq S''\subseteq E(\pi)} (-1)^{|E(\pi)\setminus S''|}\binom{x+w(S'')}{n}(y-1)^{cyc(\pi)-cyc(S'')-1}y^{cyc(S'')}((y-1)-y)\\
&=\sum_{P(\pi)\subseteq S''\subseteq E(\pi)} -(-1)^{|E(\pi)\setminus S''|}\binom{x+w(S'')}{n}(y-1)^{cyc(\pi)-cyc(S'')-1}y^{cyc(S'')}.
\end{aligned}$$

Now, taking $S'=S''$ in $B_{D'}$ and noting that $cyc'(\pi)=cyc(\pi)+1$ and $|E(\pi)|=|E'(\pi)|+1$, we see that the term in this last sum becomes

$$(-1)^{|E'(\pi)\setminus S'|}\binom{x+w(S')}{n}(y-1)^{cyc'(\pi)-c'(S)}y^{cyc'(S)} \tag{12}$$

This is exactly the term corresponding to the choice of S', with $P'(\pi) \subseteq S' \subseteq E'(\pi)$, in $B_{D'}$.

The preceding arguments show there is a bijection between the terms of B_D and the (non-canceling) terms of $B_{D'}$. Thus, we have proved $B_D(x,y) = B_{D'}(x,y)$. □

Corollary 1. *Suppose $D = (V,E,w)$ is a weighted digraph, and $D' = (V,E',w)$ is formed by adding an edge $e = (u,v)$ to E with $w(e) = 0$ for every pair $(u,v) \notin E$. Then*

$$B_{D'}(x,y) = B_D(x,y).$$

Proof. Just apply Theorem 1 recursively. □

4 An alternative form for $B_D(x,y)$.

Suppose $D = (V,E,w)$ is a weighted digraph where $E = V \times V$ (so that there may be many weight 0 edges). A *partial permutation* σ on V is an injective mapping $\sigma : U \to V$ for some subset $U \subseteq V$. There are several ways to represent σ. One is by the familiar two-line notation:

$$\sigma = \begin{pmatrix} u_1 & u_2 & \dots & u_k \\ \sigma(u_1) & \sigma(u_2) & \dots & \sigma(u_k) \end{pmatrix}$$

where u_i's denote distinct vertices of D. Here, we assume that $|U| = k$ and we will say that σ has size $|\sigma| = k$.

Another way to represent σ is to identify it with the corresponding entries in an associated matrix $M = M(D)$ indexed by elements of V. Here, the entry $M(u_i, \sigma(u_i))$ of M is just the weight $w(e_i)$ of the edge $e_i = (u_i, \sigma(u_i))$. If $S(\sigma)$ denotes the set of edges formed by σ, then it is clear that $S(\sigma)$ consists of vertex disjoint paths and cycles. Since we assume $E = V \times V$, we can abuse notation slightly by just saying that $S = S(\sigma)$ is a partial permutation, since specifying the entries in the matrix M automatically determines the actual partial permutation σ. Denote by $w(S)$ denote the sum of the weights of the edges in S and let $cyc(S)$ the number of cycles in D formed by the edges in S. Also, let $PP(V)$ denote the set of all partial permutations on V. We then have the following alternative expression for $B_D(x, y)$.

Theorem 2. *For any weighted digraph D, we have*

$$B_D(x, y) = \sum_{S \in PP(V)} \binom{x + w(S)}{n} (1 - y)^{\underline{n-|S|}} y^{cyc(S)}. \tag{13}$$

Proof. From Theorem 1, we may assume $E = V \times V$ by adding edges of weight 0 while the drop polynomial $B_D(x, y)$ remains unchanged. Consequently, for any permutation $\pi : V \to V$, we have $P(\pi) = \emptyset$. Hence, we can interchange the order of summation in (9) to obtain:

$$\begin{aligned} B_D(x, y) &= \sum_{\pi} \sum_{S \subseteq E(\pi)} (-1)^{|E(\pi) \setminus S|} \binom{x + w(S)}{n} (y - 1)^{cyc(\pi) - cyc(S)} y^{cyc(S)} \\ &= \sum_{S \in PP(V)} \sum_{\pi} (-1)^{|E(\pi) \setminus S|} \binom{x + w(S)}{n} (y - 1)^{cyc(\pi) - cyc(S)} y^{cyc(S)} \end{aligned}$$

where π ranges over all permutations of V for which $S \subseteq E(\pi)$ so that S ranges over all partial permutations. The number of π with k cycles is given by $\left[{n \atop k} \right]$, a Stirling number of the first kind (see [9]). More generally, if $|S| = s$, then π has $n - s$ free blocks from which to form $k - cyc(S)$ cycles. Thus, there are just $\left[{n-|S| \atop k-cyc(S)} \right]$ such π with $S \subseteq E(\pi)$. A basic identity for Stirling numbers of the first kind is the following (see [9]):

$$\sum_k \begin{bmatrix} m \\ k \end{bmatrix} z^k = z^{\overline{m}} \tag{14}$$

where $z^{\overline{m}}$ denotes the rising factorial $z^{\overline{m}} = z(z + 1) \ldots (z + m - 1)$. Hence our expression for $B_D(x,y)$ becomes

$$\begin{aligned}
B_D(x,y) &= \sum_{S\in PP(V)} (-1)^{n-|S|}\binom{x+w(S)}{n}\sum_k \begin{bmatrix} n-|S| \\ k-cyc(S)\end{bmatrix}(y-1)^{k-cyc(S)}y^{cyc(S)} \\
&= \sum_{S\in PP(V)} (-1)^{n-|S|}\binom{x+w(S)}{n}(y-1)^{\overline{n-|S|}}y^{cyc(S)} \\
&= \sum_{S\in PP(V)} \binom{x+w(S)}{n}(1-y)^{\underline{n-|S|}}y^{cyc(S)}
\end{aligned}$$

which is (13). This completes the proof of Theorem 2. $\square$

As an example, consider the digraph $D=(V,E,w)$ where $V=[n]$, $E=[n]\times[n]$ and $w(e)=0$ for all $e\in E$. Since in this case the "reduced" digraph is just $I(n)$, a digraph with no edges, and we have seen that $B_{I(n)}(x,y)=x^{\underline{n}}$. Thus, by (13), we get the interesting identity:

Corollary 2.

$$\sum_{S\in PP(V)} (1-y)^{\underline{n-|S|}}y^{cyc(S)} = n!.$$

It doesn't appear obvious (to us) how to prove this directly.

5 A reduction/contraction rule for $B_D(x,y)$

In order to manipulate $B_D(x,y)$, we will first need to define the operations of *reduction* and *contraction* in a digraph $D=(V,E,w)$ (see Figures 1 and 2).

First, suppose $e=(u,v)$ is a non-loop edge of D (so $u\neq v$). The e-*reduced* digraph $D'=(V,E,w')$ has the same set of vertices and edges as D, with the only change being the weight of e is reduced by 1, i.e., $w'(e)=w(e)-1$. The e-*contracted* digraph D'' is slightly more complicated. For the vertex set of D'' we replace the two vertices u and v by a single vertex uv. Any edge

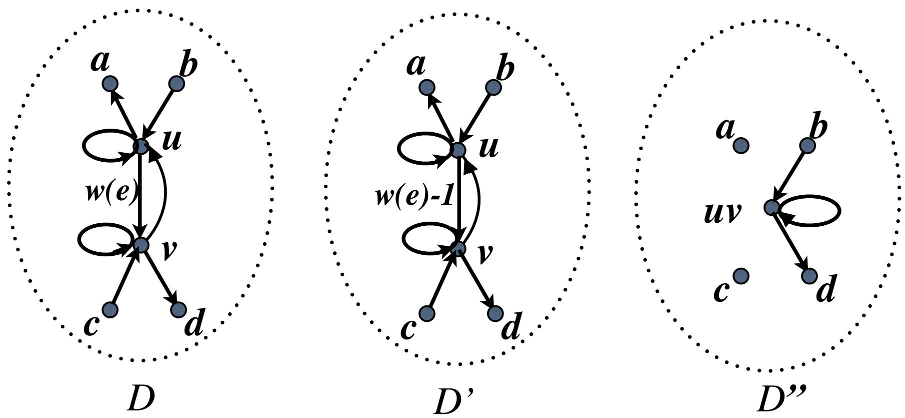


Figure 1: Reduction and contraction of a non-loop edge e in a weighted digraph

in D of the form (x,u) becomes an edge (x,uv) in D''. Also, any edge in D of the form (v,y) becomes an edge (uv,y) in D''. All other edges incident to either u or v (including loops) are deleted. If (v,u) happens to be an edge in D, it becomes a loop at uv in D''. All other edges (not incident to u or v) in D are retained in D''. No edge weights of surviving edges are changed by the contraction operation.

Next, suppose $e=(u,u)$ is a loop at u in D. Then we do the following. As before, in the e-reduced digraph D', the weight of e is reduced by 1. To form the e-*contracted* digraph D'', we simply delete the vertex u and *all* edges incident to u.

Using the representation of $B_D(x,y)$ in (13), we will now show that drop polynomials obey a reduction/contraction rule.

Theorem 3. *Let D be a weighted digraph and let e be an edge of D. Denote the e-reduced and e-contracted digraphs by D' and D'', respectively.*
(i) If e is a non-loop edge then

$$B_D(x,y) = B_{D'}(x,y) + B_{D''}(x+w(e)-1,y); \tag{15}$$

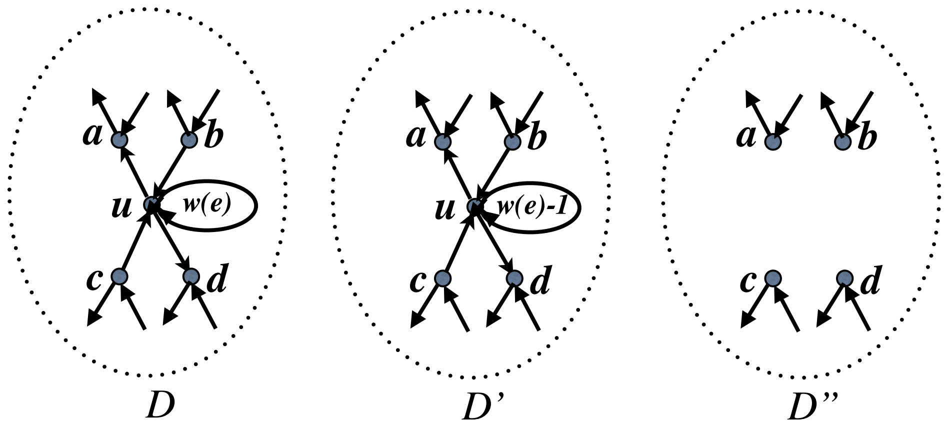


Figure 2: Reduction and contraction of a loop edge e in a weighted digraph

(ii) If e is a loop then

$$B_D(x, y) = B_{D'}(x, y) + yB_{D''}(x + w(e) - 1, y). \tag{16}$$

Proof. Let w, w', w'' denote the edge weights in D, D', D'', respectively. We first consider the case that $e = (u, v)$ is a non-loop edge. From the definition (13), we have

$$
\begin{aligned}
B_D(x,y) &= \sum_{S \in PP(V)} \binom{x+w(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} \\
&= \sum_{e \notin S} \binom{x+w(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} + \sum_{e \in S} \binom{x+w(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} \\
&= \left(\sum_{e \notin S} \binom{x+w(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} + \sum_{e \in S} \binom{x+w'(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} \right) \\
&- \sum_{e \in S} \binom{x+w'(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} + \sum_{e \in S} \binom{x+w(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} \\
&= B_{D'}(x,y) - \sum_{e \in S} \binom{x+w'(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} \\
&+ \sum_{e \in S} \binom{x+w(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)}
\end{aligned}
$$

since for $e \notin S$, we have $w(S) = w'(S)$ for any partial permutation S. We now use the fact that if $e \in S$, we have $w'(S) = w(S) - 1$. Hence

$$
\begin{aligned}
B_D(x,y) &= B_{D'}(x,y) - \sum_{e \in S} \binom{x+w(S)-1}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} \\
&+ \sum_{e \in S} \binom{x+w(S)}{n} (1-y)^{\underline{n-|S|}} y^{c(S)} \\
&= B_{D'}(x,y) + \sum_{e \in S} \binom{x+w(S)-1}{n-1} (1-y)^{\underline{n-|S|}} y^{c(S)} \\
&= B_{D'}(x,y) + \sum_{S''} \binom{x+w(S'')+w(e)-1}{n-1} (1-y)^{\underline{n-1-|S''|}} y^{c(S'')} \\
&= B_{D'}(x,y) + B_{D''}(x+w(e)-1,y)
\end{aligned}
$$

as desired.

Next, we consider the case that $e = (u,u)$ is a loop. The first part of the derivation is very similar to the preceding case and we won't repeat it. For the second part, we have

$$\begin{aligned}
B_D(x,y) &= \sum_{e\notin S}\binom{x+w(S)}{n}(1-y)^{\underline{n-|S|}}y^{c(S)} + \sum_{e\in S}\binom{x+w(S)}{n}(1-y)^{\underline{n-|S|}}y^{c(S)}\\
&= B_{D'}(x,y) - \sum_{e\in S}\binom{x+w'(S)}{n}(1-y)^{\underline{n-|S|}}y^{c(S)}\\
&+ \sum_{S}\binom{x+w(S)}{n}(1-y)^{\underline{n-|S|}}y^{c(S)}\\
&= B_{D'}(x,y) - \sum_{\substack{S''\\ S=e\cup S''}}\binom{x+w''(S)+w(e)-1}{n}(1-y)^{\underline{n-1-|S''|}}y^{c(S'')+1}\\
&+ \sum_{\substack{S''\\ S=e\cup S''}}\binom{x+w(S'')+w(e)}{n}(1-y)^{\underline{n-1-|S''|}}y^{c(S'')+1}\\
&= B_{D'}(x,y) + y\sum_{S''}\binom{x+w(S'')+w(e)-1}{n-1}(1-y)^{\underline{n-1-|S''|}}y^{c(S'')}\\
&= B_{D'}(x,y) + yB_{D''}(x+w(e)-1,y).
\end{aligned}$$

This completes the proof of Theorem 3. □

6 Relating the drop polynomial to the cover polynomial

As mentioned in the introduction, the cover polynomial $C_D(x,y)$ for a weighted digraph $D=(V,E,w)$ is defined by (6). In the case that D is *simple*, that is, $w(e)=1$ for every $e\in E$, then (6) reduces to (5):

$$C_D(x,y)=\sum_{i,j}c_D(i,j)x^{\underline{i}}y^j,$$

where we recall that $c_D(i,j)$ counts the number of path/cycle covers of D consisting of i paths and j cycles.

It was shown in [5] that for $e\in E$, if $D\setminus e$ denotes the digraph obtained from the simple digraph D by *deleting* the edge e, and D/e denotes the corresponding e-contracted digraph then we have the deletion/contraction recurrences:

(i) If e is a non-loop edge then

$$C_D(x,y) = C_{D\setminus e}(x,y) + C_{D/e}(x,y);$$

(ii) If e is a loop then

$$C_D(x,y) = C_{D\setminus e}(x,y) + yC_{D/e}(x,y).$$

Also, it is easy to see that $C_{I(n)} = x^{\underline{n}}$ where $I(n)$ denotes the n-vertex digraph having no edges.

However, these are the same recurrences that $B_D(x,y)$ satisfies for simple digraphs D, since every edge $e \in E$ has $w(e) = 1$, so that when we reduce its weight by 1, it becomes a weight 0 edge, which we know by Theorem 1 we can remove. Thus, we have

Theorem 4. *For simple digraphs $D = (V,E)$, we have*

$$B_D(x,y) = C_D(x,y).$$

Proof. The proof precedes by induction on the number of edges using the reduction/contraction rule in Theorem 3. We have seen that for the base case $D = I(n)$ with no edges:

$$B_{I(n)}(x,y) = x^{\underline{n}} = C_{I(n)}(x,y).$$

Since $B_D(x,y)$ and $C_D(x,y)$ satisfy the same recurrences for reduction and contraction, then they are equal for all simple digraphs D. $\square$

For digraphs with arbitrary edge weights, the deletion/contraction rules for C_D are the following, where $D \setminus e$ and D/e are defined above (see [6]):

(i) If e is a non-loop edge then

$$C_D(x,y) = C_{D\setminus e}(x,y) + w(e)C_{D/e}(x,y);$$

(ii) If e is a loop then

$$C_D(x,y) = C_{D\backslash e}(x,y) + y\ w(e)C_{D/e}(x,y).$$

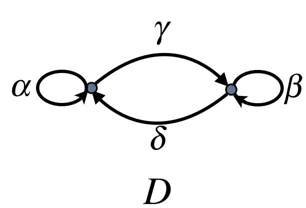
Thus, the deletion/contraction recurrences for $C_D(x,y)$ and the reduction/contraction recurrences for $B_D(x,y)$ are quite different for general weighted digraphs. However we have seen in Theorem 4 that $B_D(x,y)$ and $C_D(x,y)$ are equal when D is a *simple* digraph. However, this is not the only time they can be equal. We show an example in Figure 1 for a general weighted digraph D having 2 vertices. A little computation shows:

$$\begin{aligned} B_D(x,y) = {} & x^2 + ((\alpha+\beta)y + \gamma + \delta - 1)x + \alpha\beta y^2 + ((1/2)\alpha^2 + (1/2)\beta^2 + \gamma\delta \\ & -(1/2)\alpha - (1/2)\beta)y + (1/2)\gamma^2 + (1/2)\delta^2 - (1/2)\gamma - (1/2)\delta, \\ C_D(x,y) = {} & x^2 + ((\alpha+\beta)y + \gamma + \delta - 1)x + \alpha\beta y^2 + \gamma\delta y. \end{aligned}$$

Taking the difference we find

$$B_D(x,y) - C_D(x,y) = (1/2)((\alpha(\alpha-1) + \beta(\beta-1))y + \gamma(\gamma-1) + \delta(\delta-1)).$$

Certainly this difference is 0 if each of α, β, γ and δ is either 0 or 1. However, it is also 0 when $\alpha = 2/5, \beta = 6/5, \gamma = -1/5, \delta = 3/5$, for example.



$$\begin{bmatrix} \alpha & \gamma \\ \delta & \beta \end{bmatrix}$$

M

Figure 3: A small general weighted digraph D and its associated matrix M.

7 A reciprocity theorem

The goal here is to prove a reciprocity theorem for weighted digraphs. Let $J(n)$ denote the n-by-n matrix of all 1's.

Theorem 5. *For a weighted digraph D on n vertices, we have*

$$B_{J(n)-D}(x,y) = (-1)^n B_D(-x-y,y). \tag{17}$$

Proof. It follows from the alternative definition of $B_D(x,y)$ in Theorem 2 that it is a polynomial in x, y and the weights $w(e), e \in E$. Thus, to prove the reciprocity theorem in (17) for general digraphs, it suffices to prove it for digraphs having non-negative integer weights. This will then imply that (17) holds for arbitrary weights. We will proceed by induction on the number of edges in D. The base case for $I(n)$ holds since the reciprocity theorem holds for the cover polynomial $C_D(x,y)$ for simple digraphs (see [6] for a detailed proof) and $C_D(x,y) = B_D(x,y)$ for simple digraphs D as shown in Theorem 4.

Suppose $e = (u,v)$ is an edge in D. We first consider the case that e is a non-loop edge. We apply the reduction/contraction rule with D' and D'' as given in Theorem 3.

$$\begin{aligned} B_D(x,y) &= B_{D'}(x,y) + B_{D''}(x+w(e)-1,y) \\ &= (-1)^n B_{J(n)-D'}(-x-y,y) + (-1)^{n-1} B_{J(n-1)-D''}(-x-y+w(e)-1,y). \end{aligned}$$

We now apply the reduction/contraction rule (15) for $J(n)-D'$ using the edge e. Note that the e-reduction of $J(n)-D'$ is just $J(n)-D$. Furthermore, the e-contraction of $J(n)-D'$ is $J(n-1)-D''$. Therefore,

$$\begin{aligned} B_D(x,y) &= (-1)^n B_{J(n)-D'}(-x-y,y) + (-1)^{n-1} B_{J(n-1)-D''}(-x-y+w(e)-1,y) \\ &= (-1)^n \Big(B_{J(n)-D}(-x-y,y) + B_{J(n-1)-D''}(-x-y+w(e)-1,y) \Big) \\ &\quad + (-1)^{n-1} B_{J(n-1)-D''}(-x-y+w(e)-1,y) \\ &= (-1)^n B_{J(n)-D}(-x-y,y) \end{aligned}$$

as desired.

For the case that e is a loop, the proof is quite similar and is omitted. $\square$

An immediate consequence of (17) is the following interesting identity:

Corollary 3.

$$B_{J(n)}(x,y) = \sum_{S \in PP(v)} \binom{x+|S|}{n} (1-y)\underline{^{n-|S|}} y^{cyc(S)} = (x+y)^{\overline{n}},$$

8 Concluding remarks

There are many questions that are suggested by the preceding results. For example, although $B_D(x,y)$ and $C_D(x,y)$ are quite different when evaluated on general weighted digraphs, they agree when D is simple. Can those D for which $B_D(x,y) = C_D(x,y)$ be characterized? The fact that $B_D(x,y)$ and $C_D(x,y)$ both satisfy the same reciprocity formula for general D (see (7)) suggests that something more fundamental is responsible for this behavior.

It would be of interest to understand the computational complexity of evaluating $B_D(x,y)$. For example, for simple D, the coefficient of x^0y^1 term is just the number of Hamiltonian cycles in D, something which is known to be #P-hard to compute. (This follows from the fact that $B_D(x,y) = C_D(x,y)$ for simple digraphs D.) How hard is it to evaluate $B_D(x,y)$ at specific points in the (x,y) plane? For example,in the case that D is simple, $B_D(0,0) = 0$. However, for general weighted D, even the value of $B_D(0,0)$ does not seem to be easy to compute.

For the case of the Tutte polynomial, we mention the following result of Jaeger, Vertigan and Welsh [10]:

Theorem 6. *The problem of evaluating the Tutte polynomial $T_G(x,y)$ of a general graph G at a point (a,b) is #P-hard except when (a,b) is on the special hyperbola $(x-1)(y-1) = 1$ or when (a,b) is one of the eight special points $(1,1), (-1,-1), (0,-1), (-1,0), (i,-i), (-i,i), (j,j^2)$ and (j^2,j), where $j = e^{\frac{2\pi i}{3}}$. In each of these exceptional cases, the evaluation can be done in polynomial time.*

In the case of the cover polynomial $C_D(x,y)$ (see [5]), the situation is even worse! In this case, Bläser and H. Dell [1] have shown that it is #P-hard to evaluate $C_D(x, y)$ for general digraphs (with multiple edges and loops) at a point (a, b) unless (a, b) is one of the *three* special points $(0, 0), (0, -1)$ and $(1, -1)$ (and for these three points, the computation can be done in polynomial time). Presumably, the same result also may hold when restricted to simple digraphs D.

We point that the results for weighted digraphs presented in this paper can be restated entirely in terms of matrices with entries in some commutative ring $\mathbf{R}$ with identity. For if $D = (V, E, w)$ is our digraph on n vertices, then as we have mentioned earlier we can define a matrix $M = M(D)$ which is indexed by the elements of V. For $u, v \in V$, the (u, v) entry of M is $M(u, v) = w(e)$ if $e = (u, v) \in E$, and is 0 if $(u, v) \notin E$. In terms of matrix operations on the matrix $M = M(D)$, we can define the reduction and contraction rules in the same way as those for weighted digraphs. Then the same reciprocity theorems for digraphs (i.e., (3) and (17)) apply to the corresponding matrix drop polynomial $B_M(x, y)$ as well.

In another direction, it would be quite interesting to extend many of the known results for permutation statistics (e.g., see [13] or [11]) to the case of partial permutations. Some results in this direction have recently appeared in [4], for example. However, even the most basic questions concerning partial permutations remain unanswered at present. For example, suppose we let $N_n(s, c)$ denote the number of partial permutations on $[n]$ which have size s and contain c cycles. That is,

$$N_n(s, c) = |\{S \subseteq [n] : S \text{ is a partial permutation}, |S| = s, cyc(S) = c\}|.$$

When $s = n$ then $N_n(n, c)$ is just equal to the familiar (unsigned) Stirling number $\left[{n \atop c}\right]$ (see [9]). For example, $\left[{n \atop 1}\right] = (n-1)!$ and $\left[{n \atop 2}\right] = (n-1)!(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n-1})$. However, for $0 < s < n$, it is not hard to show that $N_n(s, 0) = \binom{n}{s}(n-1)^{\underline{s}}$ and $N_n(s, 1) = \binom{n}{s}(n-1)^{\underline{s}}(\frac{1}{n-s} + \frac{1}{n-s+1} + \ldots + \frac{1}{n-1})$. It is possible to derive explicit expressions for $N_n(s, 2)$ and $N_n(s, 3)$ but the formulas seem to be getting increasingly unpleasant! Is there nice way of expressing $N_n(s, k)$ in general?

Of course, since there are just $\binom{n}{s} n^{\underline{s}}$ possible partial permutations on $[n]$ of size s, then

$$\sum_{0 \le c \le n} N_n(s,c) = \binom{n}{s} n^{\underline{s}}.$$

However, the following identity (equivalent to Corollary 2) seems less obvious.

$$\sum_{s,c} N_n(s,c)(1-y)^{\underline{n-s}} y^c = n!.$$

If the vertex set V of D is totally ordered, say $V = [n]$, we can define $exc(S)$, the number of *exceedences* of a partial permutation S as follows. Namely, define $exc(S)$ to be the number of indices i such that $\sigma(i) > i$ for the associated mapping $\sigma : [n] \to [n]$. From the matrix perspective, this is the number of entries of S which lie above the diagonal of M. How many partial permutations S of size k have $exc(S) = d$? For the usual case of permutations of $[n]$, these numbers are the usual Eulerian numbers (see [9]). Of course, the same questions can be asked when V is just partially ordered, or more generally, when D is an arbitrary simple digraph. There is clearly much more to do.

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