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# On a conjecture of Erdös in additive number theory

by

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**1. Introduction.** Let $t$ and $a$ be real numbers and let $S_t(a)$ denote the sequence $(s_1, s_2, \ldots)$ defined by $s_n = [ta^n]$ (where [ ] denotes the greatest integer function). It was conjectured by Erdös several years ago that if $t > 0$ and $1 < a < 2$ then every sufficiently large integer $n$ can be expressed as $n = \sum_{k=1}^{\infty} \varepsilon_k s_k$ where $\varepsilon_k = 0$ or 1 and all but a finite number of the $\varepsilon_k$ are 0. In general, a sequence of integers which has this property is said to be *complete* and if every positive integer is so expressible then the sequence is said to be *entirely complete*. While the additive structure of $S_t(a)$ is far from being completely understood at present, it is the object of this paper to shed some light on this question. In particular, the set $T$ of all points $(t, a)$ of the unit square $S = \{(t, a): 0 < t < 1, 1 < a < 2\}$ for which $S_t(a)$ is complete will be determined. It will be seen $T$ has an area of approximately 0.85.

**2. Preliminary remarks.** If $A = (a_1, a_2, \ldots)$ is a sequence of integers then $P(A)$ is defined to be the set of all sums of the form $\sum_{k=1}^{\infty} \varepsilon_k a_k$ where $\varepsilon_k = 0$ or 1 and all but a finite number of the $\varepsilon_k$ are 0. In this paper, we adopt the convention that a sum of the form $\sum_{k=a}^{b}$ is 0 for $b < a$. We now give several results which will be needed later.

THEOREM 1. (J. Folkman.) *Let $A = (a_1, a_2, \ldots)$ be a sequence of positive integers such that:*

1. $a_n + a_{n+1} \leqslant a_{n+2}$ *for* $n \geqslant 1$.

2. *There exist $m \geqslant 0$ and $r \geqslant 0$ such that $m \notin P(A)$ and*

$$\sum_{k=1}^{r} a_k < m < a_{r+2}.$$

*Then $A$ is not complete.*

Proof. By hypothesis we have

$$\sum_{k=1}^{r+2} a_k = \sum_{k=1}^{r} a_k + a_{r+1} + a_{r+2} < m + a_{r+3} < a_{r+2} + a_{r+3} \leqslant a_{r+4}.$$

Therefore, $m + a_{r+3} \notin P(A)$ and $\sum_{k=1}^{r+2} a_k < m + a_{r+3} < a_{r+4}$. Since we can now apply the same argument with $m$ and $r$ replaced by $m + a_{r+3}$ and $r+2$, respectively, then by induction on $r$ we conclude that $A$ is not complete.

LEMMA 1. *Let $A = (a_1, a_2, \ldots)$ be a nondecreasing sequence of positive integers. Then the following statements are equivalent*:

1. *$A$ is entirely complete.*
2. *For all $n \geqslant 0$, $\sum_{k=1}^{n} a_k \geqslant a_{n+1} - 1$.*
3. *For all $n \geqslant 0$, $\sum_{a_k \leqslant n} a_k \geqslant n$.*
4. *For all $n \geqslant 0$, $a_{n+1} - 1 \in P(A)$.*

Proof. 1 $\Rightarrow$ 4.: This is immediate.

4. $\Rightarrow$ 3. If there is an $n$ such that $\sum_{a_k \leqslant n} a_k < n$ then there is a least $r$ such that $a_r > n$. Thus,

$$a_r - 1 \geqslant n > \sum_{a_k \leqslant n} a_k = \sum_{k=1}^{r-1} a_k \quad \text{and hence} \quad a_r - 1 \notin P(A),$$

contradicting 4.

3. $\Rightarrow$ 2. If there is an $n$ such that $\sum_{k=1}^{n} a_k < a_{n+1} - 1$, then $\sum_{a_k \leqslant a_{n+1}} a_k = \sum_{k=1}^{n} a_k < a_{n+1} - 1$ which contradicts 3.

2. $\Rightarrow$ 1. This is a result of J. L. Brown [1] and the proof of Lemma 1 is completed.

LEMMA 2. *Let $A = (a_1, a_2, \ldots)$ be a nondecreasing sequence of positive integers and suppose there exists an $r$ such that*:

1. $\sum_{k=1}^{m} a_k \geqslant a_{m+1} - 1$ *for* $0 \leqslant m \leqslant r$.
2. $a_{m+1} \leqslant 2a_m$ *for* $m \geqslant r+1$.

*Then $A$ is entirely complete.*

Proof. For any $c > 0$ we have

$$\sum_{k=1}^{r+c} a_k = \sum_{k=1}^{r} a_k + \sum_{k=r+1}^{r+c} a_k \geqslant a_{r+1} - 1 + \sum_{k=r+1}^{r+c} (a_{k+1} - a_k) = a_{r+c+1} - 1$$

and hence by Lemma 1, $A$ is entirely complete.

We next state three lemmas whose proofs are immediate and will be omitted.

LEMMA 3. *If* $t > 0$ *and* $a \geqslant (1+\sqrt{5})/2$ *then*

$$[ta^{n+2}] \geqslant [ta^{n+1}] + [ta^n] \quad \text{for} \quad n \geqslant 1.$$

LEMMA 4. *If* $t > 0$ *and* $1 < a < 2$ *then:*

1. $[ta^{n+1}] \leqslant 2[ta^n]$ *for* $[ta^n] \geqslant (a-1)/(2-a)$.
2. $[ta^{n+1}] \leqslant 2[ta^n]+1$ *for* $n \geqslant 0$.

LEMMA 5. $(1+x)^y \geqslant 1+yx$ *for* $x \geqslant -1$ *and* $y \geqslant 1$ (cf. Korovkin [2]).

**3. The structure of** $T$. We first note that for $0 < t < 1$ and $1 < a < 2$ we have

$$S_{t/a}(a) = ([ta/a], [ta^2/a], \ldots) = ([t], [ta], [ta^2], \ldots) = (0, [ta], [ta^2], \ldots).$$

Thus, $P(S_{t/a}(a)) = P(S_t(a))$ and consequently if we can determine $P(S_t(a))$ for $1/a \leqslant t < 1$ then we immediately know $P(S_t(a))$ for $0 < t < 1$. For $1/a \leqslant t < 1$ and $1 < a < 2$ we have

$$1 = [a/a] \leqslant [ta] = s_1 \leqslant [a] = 1, \quad \text{i.e.,} \quad s_1 = 1.$$

THEOREM 2. *If* $0 < t < 1$ *and* $1 < a \leqslant 5^{1/3}$ *then* $S_t(a)$ *is entirely complete.*

Proof. By the preceding remark, it suffices to prove the theorem for $1/a \leqslant t < 1$. Thus $s_1 = 1$ and $s_2 = [ta^2] \leqslant [5^{2/3}] = 2$. Since $ta^3 < a^3 \leqslant 5$, then $s_3 \leqslant 4$. The only possible ways that $S_t(a)$ can start are as follows:

$$S_t(a) = (1, 2, m, \ldots) \quad \text{for} \quad m \leqslant 4 \quad (\text{since } s_3 \leqslant 4),$$
$$S_t(a) = (1, 1, m, \ldots) \quad \text{for} \quad m \leqslant 3 \quad (\text{by Lemma 4}).$$

By Lemma 2, for $s_n \geqslant 3 > (5^{1/3}-1)/(2-5^{1/3}) \geqslant (a-1)/(2-a)$ we have $s_{n+1} \leqslant 2s_n$. Thus, if $k$ is the least integer such that $s_k \geqslant 3$ then by Lemma 4 we must have $\sum_{i=1}^{r} s_i \geqslant s_{r+1}-1$ for $0 \leqslant r \leqslant k-1$. Hence, by Lemma 2, $S_t(a)$ is entirely complete for $1/a \leqslant t < 1$ and $1 < a \leqslant 5^{1/3}$. Therefore $S_t(a)$ is entirely complete for $0 < t < 1$ and $1 < a \leqslant 5^{1/3}$ and the proof is completed.

THEOREM 3. *Suppose* $0 < t < 1$ *and* $1 < a < 2$. *Then* $S_t(a)$ *is complete if and only if* $S_t(a)$ *is entirely complete.*

Proof. As before it suffices to prove the theorem for $1/a \leqslant t < 1$. For $1 < a \leqslant 5^{1/3}$ this result is established in Theorem 2. Let $5^{1/3} < a < 2$ and suppose $S_t(a)$ is not entirely complete. By Lemma 1 there is a least

$m$ such that $\sum_{s_k \leqslant m} s_k < m$. Therefore $m < s_{n+1} \leqslant s_{n+2}$ where $s_n$ is the greatest element of $S_t(a)$ which does not exceed $m$. (Note that $m \notin P(S_t(a))$). Since $1 = s_1 \in P(S_t(a))$ then $m \geqslant 2$, $n \geqslant 1$ and $\sum_{k=1}^{n} s_k < m < s_{n+2}$. But $a > 5^{1/3} > (1+\sqrt{5})/2$ and thus, by Lemma 3, $s_{n+2} \geqslant s_{n+1} + s_n$ for $n \geqslant 1$. By applying Theorem 1 we see that $S_t(a)$ is not complete. Since an entirely complete sequence is always complete, then the theorem is proved.

Now, let $d_n = s_{n+1} - 2s_n$ for $n \geqslant 1$ and let $D_n = \sum_{k=1}^{n} d_k$ for $n \geqslant 0$. Lemma 4 implies that $d_n \leqslant 1$ while it is easily shown for $n \geqslant 0$ that $s_{n+1} - \sum_{k=1}^{n} s_k = 1 + D_n$. For the following four lemmas we shall assume that $1/a \leqslant t < 1$ and $1 < a < 2$. From these lemmas the structure of $T$ will follow immediately.

Lemma 6. *$S_t(a)$ is complete if and only if $D_n \leqslant 0$ for all $n \geqslant 0$.*

Proof. This follows at once from Theorem 3 and Lemma 1.

Lemma 7. *If $d_n \leqslant -2$ then $d_{n+k} \leqslant 0$ for all $k \geqslant 0$.*

Lemma 8. *If $d_n = -1$ and $d_{n+1} \leqslant 0$ then $d_{n+k} \leqslant 0$ for $k \geqslant 3$.*

Lemma 9. *If $d_n = -1$, $d_{n+1} = 1$ and $d_{n+2} \leqslant 0$ then $d_{n+k} \leqslant 0$ for $k \geqslant 4$.*

The proofs of these lemmas are straightforward and we shall give only a proof of Lemma 8, which is typical.

Proof of Lemma 8. By hypothesis we have $s_{n+1} = 2s_n - 1$. If $d_{n+1} = 0$ then $s_{n+2} = 2s_{n+1} = 4s_n - 2$. Thus,

$$ta^{n+2} < s_{n+2} + 1 = 4s_n - 1 \leqslant 4ta^n - 1$$

and

$$a^2 < 4 - 1/ta^n. \tag{1}$$

If $d_{n+1} \leqslant -1$ then $s_{n+2} \leqslant 2s_{n+1} - 1 = 4s_n - 3$. Thus,

$$ta^{n+2} < s_{n+2} + 1 \leqslant 4s_n - 2 \leqslant 4ta^n - 2$$

and

$$a^2 < 4 - 2/ta^n. \tag{1'}$$

Now suppose there exists $k \geqslant 3$ such that $d_{n+k} = 1$. Then

$$\begin{aligned} ta^{n+k+1} &\geqslant s_{n+k+1} = 2s_{n+k} + 1 \\ &> 2(ta^{n+k} - 1) + 1 = 2ta^{n+k} - 1. \end{aligned} \tag{2}$$

Hence $a > 2-1/ta^{n+k}$ and consequently, by Lemma 5,

$$a^2 > (2-1/ta^{n+k})^2 = 4(1-1/2ta^{n+k})^2$$
$$\geqslant 4(1-2/2ta^{n+k}) = 4-4/ta^{n+k}.$$

There are two cases:

(i) If $d_{n+1} = 0$ then by (1) we have

$$4-\frac{1}{ta^n} > 4-\frac{4}{ta^{n+k}}$$

and therefore $a^k < 4$. Since $k \geqslant 3$, then $a^3 \leqslant a^k < 4$ and by (2) we have

$$4 > a^3 > 8(1-1/2ta^{n+k})^3 \geqslant 8(1-3/2ta^{n+k}).$$

Hence $s_{n+k} \leqslant ta^{n+k} < 3$. This is impossible however since $k \geqslant 3$ and $s_n \geqslant 1$ imply

$$s_{n+k} \geqslant s_{n+3} \geqslant 2s_{n+2}-1 = 8s_n-5 \geqslant 3$$

(since by Lemma 7 we cannot have $d_{n+2} \leqslant -2$ and $d_{n+k} = 1$) and case (i) is completed.

(ii) If $d_{n+1} \leqslant -1$ then by (1′) we have

$$4-\frac{2}{ta^n} > 4-\frac{4}{ta^{n+k}}$$

and therefore $a^k < 2$. Since $k \geqslant 3$, then $a^3 \leqslant a^k < 2$ and by (2) we have

$$2 > a^3 > 8(1-1/2ta^{n+k})^3 \geqslant 8(1-3/2ta^{n+k}).$$

Hence $ta^{n+k} < 2$. But $d_{n+k} = 1$ implies $s_{n+k+1} = 2s_{n+k}+1 \geqslant 3$. Thus $2a > ta^{n+k+1} \geqslant 3$ and therefore $a > 3/2$. However, this contradicts the previous conclusion that $a^3 < 2$ and case (ii) is completed.

Thus, we have shown that there cannot exist $n \geqslant 1$ and $k \geqslant 3$ such that $d_n = -1$, $d_{n+1} \leqslant 0$ and $d_{n+k} > 0$. This completes the proof.

We can now prove the basic

THEOREM 4. *Suppose* $1/a \leqslant t < 1$ *and* $1 < a < 2$. *Then* $S_t(a)$ *is not complete if and only if for some* $n \geqslant 0$ *one of the following holds*:

1. $d_n = 1$, $d_m = 0$ *for* $m < n$.
2. $d_n = -1$, $d_{n+1} = 1$, $d_{n+2} = 1$, $d_m = 0$ *for* $m < n$.
3. $d_n = -1$, $d_{n+1} = 1$, $d_{n+2} = 0$, $d_{n+3} = 1$, $d_m = 0$ *for* $m < n$.

Proof. This theorem follows at once from Lemmas 6, 7, 8, and 9 by considering the first occurrence of a nonzero $d_k$.

Let $A_n$, $B_n$ and $C_n$ denote the sets of all points $(t, a)$ of the $t-a$ plane for which $S_t(a)$ falls into cases 1, 2 and 3, respectively, of Theorem 4. These sets are characterized by the following theorem.

THEOREM 5. (I). *$(t, a) \epsilon A_n$ if and only if*

1. $ta^n < 2^{n-1}+1$.
2. $ta^{n+1} \geqslant 2^n+1$.
3. $1 < a < 2$.
4. $1/a \leqslant t < 1$.

(II). *$(t, a) \epsilon B_n$ if and only if*

1. $ta^{n+1} < 2^n$.
2. $ta^n \geqslant 2^{n-1}$.
3. $ta^{n+3} \geqslant 2^{n+2}-1$.

(III). *$(t, a) \epsilon C_n$ if and only if*

1. $ta^{n+1} < 2^n$.
2. $ta^{n+3} < 2^{n+2}-1$.
3. $ta^{n+4} \geqslant 2^{n+3}-3$.
4. $ta^n \geqslant 2^{n-1}$.

Proof. (I). From the definition of $A_n$ we know that $(t, a) \epsilon A_n$ if and only if $1/a \leqslant t < 1$, $1 < a < 2$, $d_n = 1$ and $d_m = 0$ for $m < n$. In this case we have

$$S_t(a) = (1, 2, 2^2, \ldots, 2^{n-1}, 2^n, 2^{n+1}+1, \ldots)$$

and consequently $ta^n < s_n+1 = 2^{n-1}+1$ and $ta^{n+1} \geqslant s_{n+1} = 2^n+1$ which establishes the necessity of conditions 1-4. To show sufficiency assume conditions 1-4 hold. Then $ta^{n+1} \geqslant 2^n+1$ implies $ta^k \geqslant 2^{k-1}$ for $1 \leqslant k \leqslant n$. Also from $ta^n < 2^{n-1}+1$ we have $(2^{n-1}+1)a > ta^{n+1} \geqslant 2^n+1$ and thus

$$a > \frac{2^n+1}{2^{n-1}+1} > \frac{2^{n-1}+1}{2^{n-2}+1} > \frac{2^{n-2}+1}{2^{n-3}+1} > \ldots$$

Therefore, $ta^k < 2^{k-1}+1$ for $1 \leqslant k \leqslant n$. Finally, since $ta^n < 2^{n-1}+1$ and $a < 2$ imply $ta^{n+1} < 2^n+2$ then from conditions 1-4 we see that $S_t(a) = (1, 2, 2^2, \ldots, 2^{n-1}, 2^n, 2^{n+1}+1, \ldots)$ and consequently $(t, a) \epsilon A_n$. This proves (I). The proofs of (II) and (III) are quite similar and will be omitted.

It is now an easy matter to relax the restriction $1/a \leqslant t$. For any $0 < t < 1$ and $1 < a < 2$ there is a unique $m$ such that $1/a \leqslant ta^m < 1$. We have already noted that $S_t(a)$ is complete if and only if $S_{ta^m}(a)$ is complete. Hence each sequence $S_t(a)$ for $1/a \leqslant t < 1$ which is not complete generates a family of sequences $S_{t/a^m}(a)$, $m = 1, 2, \ldots$, which are not complete. Thus if we let $A_n^{(m)}$ denote the set $\{(t/\alpha^m, \alpha): (t, \alpha) \in A_n\}$ for $m = 0, 1, 2, \ldots$ (so that $A_n^{(0)} = A_n$) with $B_n^{(m)}$ and $C_n^{(m)}$ defined similarly then we have

THEOREM 6. *Suppose* $0 < t < 1$ *and* $1 < \alpha < 2$. *Then* $S_t(\alpha)$ *is not complete if and only if*

$$(t, \alpha) \in \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} A_n^{(m)} \cup B_n^{(m)} \cup C_n^{(m)}.$$

The complement of this set with respect to the unit square $S$ is just the set $T$ of all points $(t, \alpha)$ in $S$ for which $S_t(\alpha)$ is complete. A portion of $T$ is graphically represented in Fig. 1. It is not difficult to verify that each of $A_n^{(m)}$, $B_n^{(m)}$ and $C_n^{(m)}$ is nonempty and that the area of $T$ is approximately 0.85.

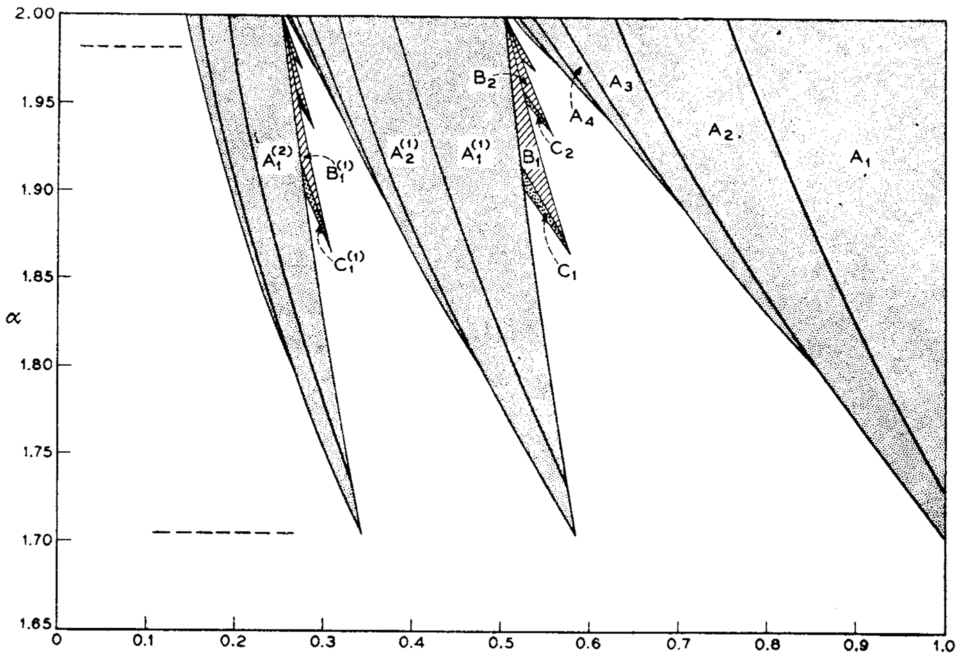


Fig. 1

**4. Concluding remarks.** In general, it seems to be a difficult problem to determine all the points $(t, \alpha)$ with $t > 1$ for which $S_t(\alpha)$ is complete. It follows from Theorem 1 that $S_t(\alpha)$ is not complete for $\alpha \geq \max\left(\frac{2}{t}, \frac{1+\sqrt{5}}{2}\right)$. On the other hand, it is not difficult to show that $S_t(\alpha)$ is complete for $t = 2^{k/2}$ and $\alpha = 2^{1/2}$ ($k$ an arbitrary integer). It would not unreasonable to conjecture that $S_t(a)$ is complete for $t > 0$ and $1 < a < (1+\sqrt{5})/2$. However, even for the case of $t = (3/2)^k$ and $a = 3/2$ it is not known if any terms of $S_t(a)$ are odd for $k$ sufficiently large.

**References**

[1] J. L. Brown, *Note on complete sequences of integers*, Amer. Math. Monthly 68 (1961), pp. 557-560.

[2] P. P. Korovkin, *Inequalities*, London 1961.

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