A PROPERTY OF FIBONACCI NUMBERS

R. L. GRAHAM
Bell Telephone Laboratories, Inc., Murray Hill, N. J.

1. INTRODUCTION

Let $A = (a_1, a_2, \cdots)$ denote a (possibly finite) sequence of integers. We shall let $P(A)$ denote the set of all integers of the form $\sum_{k=1}^{\infty} \epsilon_k a_k$ where ϵ_k is 0 or 1. If all sufficiently large integers belong to $P(A)$ then A is said to be complete. For example, if $F = (F_1, F_2, \cdots)$, where F_n is the n^{th} Fibonacci number, i.e., $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, then F is complete (cf. [1]). More generally, it can be easily shown that F satisfies the following conditions:

(A) If any one term is removed from F then the resulting sequence is complete.

(B) If any two terms are removed from F then the resulting sequence is not complete.

(A simple proof of (A) is given in [1]; (B) will be proved in Section 2.)

In this paper it will be shown that a "slight" modification of F produces a rather startling change in the additive properties of F. In particular, the sequence S which has $F_n - (-1)^n$ as its n^{th} term has the following remarkable properties:

(C) If any finite subsequence is deleted from S then the resulting sequence is complete.

(D) If any infinite subsequence is deleted from S then the resulting sequence is not complete.

2. THE MAIN RESULTS

We first prove (B). Suppose F_r and F_s are removed from F to form F^* (where $r < s$). We show by induction that $F_{s+2k+1} - 1 \notin P(F^*)$ for $k = 0, 1, 2, \cdots$. We first note that the sum of all terms of F^* which do not exceed $F_{s+1} - 1$ is just

$$\sum_{k=1}^{s-1} F_k - F_r = \sum_{k=1}^{s-1} (F_{k+2} - F_{k+1}) - F_r = F_{s+1} - 1 - F_r < F_{s+1} - 1$$

and hence $F_{s+1} - 1 \notin P(F^*)$. Now assume that $F_{s+2t+1} - 1 \notin P(F^*)$ for some $t \geq 0$ and consider the integer $F_{s+2t+3} - 1$. The sum of all terms of F^* which are less than F_{s+2t+2} is just

$$\sum_{k=1}^{s+2t+1} F_k - F_r - F_s = F_{s+2t+3} - 1 - F_r - F_s < F_{s+2t+3} - 1 \quad .$$

Thus, in order to have $F_{s+2t+3} - 1 \in P(F^*)$ we must have $F_{s+2t+3} - 1 = F_{s+2t+2} + m$, where $m \in P(F^*)$. But $m = F_{s+2t+3} - F_{s+2t+2} - 1 = F_{s+2t+1} - 1$ which does not belong to $P(F^*)$ by assumption. Hence $F_{s+2t+3} - 1 \notin P(F^*)$ and proof of (B) is completed.

We now proceed to the main result of the paper.

Theorem: Let $S = (s_1, s_2, \cdots)$ be the sequence of integers defined by $s_n = F_n - (-1)^n$. Then S satisfies (C) and (D).

Proof: The proof of (D) will be given first. Let the infinite subsequence $s_{i_1} < s_{i_2} < s_{i_3} < \cdots$ be deleted from S and denote the remaining sequence by S^*. In order to prove (D) it suffices to show that

$$s_{i_n+1} - 1 \notin P(S^*) \text{ for } n \geq 4 \quad .$$

We first note that

$$s_{i_1} + s_{i_2} \geq s_1 + s_2 = 2 \quad .$$

Therefore, we have (cf. Eq. (1))

$$\sum_{\substack{j=1 \\ j \neq i_1, i_2}}^{i_n - 1} s_j < s_{i_n+1} - s_{i_1} - s_{i_2} \leq s_{i_n+1} - 2 \quad .$$

Hence, to represent $s_{i_n+1} - 1$ in $P(S^*)$ we must use some term of S^* which exceeds s_{i_n-1} (since by above, the sum of all terms of S^* not exceeding s_{i_n-1} is less than $s_{i_n+1} - 1$ for $n \geq 4$). Since s_{i_n} is missing from S^*, then the smallest term of S^* which exceeds s_{i_n-1} is s_{i_n+1} (which, of course, is greater than $s_{i_n+1} - 1$). Thus

$$s_{i_n+1} - 1 \notin P(S^*) \text{ for } n \ge 4$$

and (D) is proved.

To prove (C), let $k > 4$ and let S' denote the sequence $(s_k, s_{k+1}, s_{k+2}, \cdots)$. For non-negative integers w and x, $P(S')$ is said to have <u>no gaps of length greater than w beyond x</u> provided there do not exist $w + 1$ consecutive integers exceeding x which do not belong to $P(S')$. The proof of (C) is now a consequence of the following two lemmas.

<u>Lemma 1</u>: There exists v such that $P(S')$ has no gaps of length greater than v beyond s_k.

<u>Lemma 2</u>: If $w > 0$ and $P(S')$ has no gaps of length greater than w beyond s_h then there exists i such that $P(S')$ has no gaps of length greater than $w - 1$ beyond s_i.

Indeed, by Lemma 1 and repeated application of Lemma 2 it follows that there exists j such that $P(S')$ has no gaps of length greater than 0 beyond s_j. That is, S' is complete, which proves (C).

<u>Proof of Lemma 1:</u> First note that

$$s_{2n} + s_{2n+1} = F_{2n} - (-1)^{2n} + F_{2n+1} - (-1)^{2n+1} = F_{2n} + F_{2n+1} = F_{2n+2} = s_{2n+2} + 1.$$

Similarly,

$$\begin{aligned} s_{2n+1} + s_{2n+2} &= F_{2n+1} - (-1)^{2n+1} + F_{2n+2} - (-1)^{2n+2} \\ &= F_{2n+1} + F_{2n+2} = F_{2n+3} = s_{2n+3} - 1 . \end{aligned}$$

Also, we have

$$(1) \quad \begin{cases} s_1 + s_2 + \cdots + s_n = (F_1 + 1) + (F_2 - 1) + \cdots + (F_n - (-1)^n) \\ \quad = \sum_{j=1}^{n} F_j + \epsilon_n = \sum_{j=1}^{n} (F_{j+2} - F_{j+1}) + \epsilon_n \\ \quad = F_{n+2} - 1 + \epsilon_n \\ \quad = s_{n+2} - \epsilon_n \end{cases}$$

where

$$\epsilon_n = \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd} \end{cases} .$$

Thus

$$\sum_{j=m}^{n} s_j = s_{n+2} - s_{m+1} - \epsilon_n + \epsilon_{m-1} \quad \text{for } n \geq m .$$

Now, let $h > k + 1$ and let

$$P' = P((s_k, s_{k+1}, \cdots, s_h)) = \{p'_1, p'_2, \cdots, p'_n\}$$

where $p'_1 < p'_2 < \cdots < p'_n$. Let

$$v = \max_{1 \leq r \leq n-1} (p'_{r+1} - p'_r) .$$

Then

$$\begin{aligned} h > k + 1 > 5 &\Longrightarrow s_h \geq s_{k+1} + 2 \\ &\Longrightarrow s_h \geq s_{k+1} + \epsilon_h - \epsilon_{k+1} + 1 \\ &\Longrightarrow s_{h+2} - s_{h+1} \geq s_{k+1} + \epsilon_h - \epsilon_{k+1} \\ &\Longrightarrow s_{h+1} \leq s_{h+2} - s_{k+1} - \epsilon_h + \epsilon_{k+1} = \sum_{j=1}^{h} s_j . \end{aligned}$$

Since

$$\max_{1 \leq r \leq n-1} \left((p'_{r+1} + s_{h+1}) - (p'_r + s_{h+1}) \right) = v$$

then in

$$\begin{aligned} P'' &= P\left((s_k, \cdots, s_h, s_{h+1})\right) \\ &= P\left((s_k, \cdots, s_h)\right) \cup \left\{q + s_{h+1} : q \in P\left((s_k, \cdots, s_h)\right)\right\} \\ &= \{p''_1, p''_2, \cdots, p''_{n'}\} , \end{aligned}$$

where $p''_1 < p''_2 < \cdots < p''_{n'}$, we have

$$\max_{1 \le r \le n'-1} (p''_{r+1} - p''_r) \le v \ .$$

Similarly, since

$$h > k + 1 > 5 \Rightarrow s_{h+2} \le \sum_{j=k}^{h+1} s_j$$

then in

$$P''' = P\left((s_k, \cdots, s_{h+2})\right) = \left\{ p'''_1, p'''_2, \cdots, p'''_{n''} \right\}$$

where $p'''_1 < p'''_2 < \cdots < p'''_{n''}$, we have

$$\max_{1 \le r \le n''-1} (p'''_{r+1} - p'''_r) \le v, \text{ etc.}$$

By continuing in this way, Lemma 1 is proved.

The proof of Lemma 2 is a consequence of the following two results:

(a) For any $r \ge 0$ there exists t such that $m > t$ implies all the integers

$$s_m + y, \quad y = 0, \pm 1, \pm 2, \cdots, \pm(r - 1)$$

belong to $P(S')$.

(b) There exists r' such that for all sufficiently large h', $P(S')$ has no gaps of length greater than $w - 1$ between $s_{h'} + r'$ and $s_{h'+1} - r'$ (i.e., there do not exist w consecutive integers exceeding $s_{h'} + r'$ and less than $s_{h'+1} - r'$ which are missing from $P(S')$).

Therefore, for s_i sufficiently large, $P(S')$ has no gaps of length greater than $w - 1$ beyond s_i, which proves Lemma 2.

Proof of (a): Choose p such that

$$2p - 3 \ge k \qquad \text{and} \qquad s_{2p-2} \ge r$$

and choose n such that

$$n \ge s_{2p-2} + p \quad \text{and} \quad n \ge r + k \ .$$

Then

$$\sum_{i=n-m}^{n} s_{2i-1} + \sum_{j=2p-3}^{2n-2m-4} s_j = \sum_{i=1}^{n} s_{2i-1} - \sum_{i=1}^{n-m-1} s_{2i-1} + \sum_{j=2p-3}^{2n-2m-4} s_j$$

$$= n + \sum_{i=1}^{n} F_{2i-1} - n + m + 1 - \sum_{i=1}^{n-m-1} F_{2i-1} + \sum_{j=2p-3}^{2n-2m-4} s_j$$

$$= m + 1 + F_{2n} - F_{2n-2m-2} + s_{2n-2m-2} + 0 - s_{2p-2} - 0$$

$$= s_{2n} - (s_{2p-2} - m - 1), \text{ for } 0 \le m \le n - p - 1 .$$

Since $2p - 3 \ge k$, then all the summands used on the left-hand side are in S'. Hence, all the integers

$$s_{2n} - (s_{2p-2} - m - 1), \quad 0 \le m \le n - p - 1 ,$$

belong to $P(S')$. Since $n \ge s_{2p-2} + p$, then

$$n - p - 1 \ge s_{2p-2} - 1 .$$

Therefore, all the integers

$$s_{2n} - (s_{2p-2} - m - 1), \quad 0 \le m \le s_{2p-2} - 1 ,$$

belong to $P(S')$, i.e., all the integers

$$s_{2n} - m', \qquad 0 \le m' \le s_{2p-2} - 1 .$$

But $s_{2p-2} \ge r$, so that we finally see that all the integers

$$s_{2n} - m', \quad 0 \le m' \le r - 1 ,$$

belong to $P(S')$.

To obtain sums which exceed s_{2n}, note that for $1 \le m \le n - k$ we have

$$\sum_{j=n-m+1}^{n} s_{2j-1} + s_{2n-2m} = \sum_{j=1}^{n} s_{2j-1} - \sum_{j=1}^{n-m} s_{2j-1} + s_{2n-2m}$$

$$= n + F_{2n} - (n - m) - F_{2n-2m} + s_{2n-2m}$$

$$= m + F_{2n} - 1$$

$$= m + s_{2n} \quad .$$

Since the sums

$$\sum_{j=n-m+1}^{n} s_{2j-1} + s_{2n-2m} \quad \text{for } m = 1, 2, \cdots, n - k$$

are all elements of $P(S')$, and since $n - k \geq r$, then all the integers

$$s_{2n} + m, \quad 1 \leq m \leq r \quad ,$$

belong to $P(S')$.

Arguments almost identical to this show that for all sufficiently large n, all the integers

$$s_{2n+1} + m, \quad m = 0, \pm 1, \cdots, \pm(r - 1) \quad ,$$

belong to $P(S')$. This proves (a).

Proof of (b): We first give a definition. Let $A = (a_1, a_2, \cdots, a_n)$ be a finite sequence of integers. The point of symmetry of $P(A)$ is defined to be the number $\frac{1}{2}\sum_{k=1}^{n} a_k$. The reason for this terminology arises from the fact that if $P(A)$ is considered as a subset of the real line, then $P(A)$ is symmetric about the point $\frac{1}{2}\sum_{k=1}^{n} a_k$. For we have

$$p = \sum_{k=1}^{n} \epsilon_k a_k \in P(A) \Longleftrightarrow \sum_{k=1}^{n} (1 - \epsilon_k) a_k = \sum_{k=1}^{n} a_k - p \in P(A)$$

and the points p and $\sum_{k=1}^{n} a_k - p$ are certainly equidistant from $\frac{1}{2}\sum_{k=1}^{n} a_k$.

Now note that if r is sufficiently large then

$$s_{r-1} > 3 > -s_{k+1} + 3 ,$$

$$s_{r+1} - s_r > -s_{k+1} + 2 ,$$

$$s_r + 1 - s_{k+1} < s_{r+1} - 1 ,$$

$$s_{r+2} - s_{r+1} - s_{k+1} < s_{r+1} + \epsilon_r - \epsilon_{k+1} ,$$

and

$$s_{r+2} - s_{k+1} < 2s_{r+1} + \epsilon_r - \epsilon_{k+1} .$$

Therefore

$$\frac{1}{2}\sum_{j=k}^{r} s_j = \frac{1}{2}(s_{r+2} - s_{k+1} - \epsilon_r + \epsilon_{k+1}) < s_{r+1}$$

and

$$\frac{1}{2}(s_{r+2} - s_{k+1} - \epsilon_r + \epsilon_{k+1}) > s_h$$

for all sufficiently large r. In other words, for all sufficiently large r, the point of symmetry of $P((s_k,\cdots,s_r))$ lies between s_h and s_{r+1}. By hypothesis no gaps of length greater than w occur in $P(S')$ beyond s_h. Since $h > k > 4$ implies

$$s_h < s_{h+1} < s_{h+2} < \cdots ,$$

then no gaps of length greater than w can occur in $P((s_k,\cdots,s_r))$ between s_h and s_{r+1}. (For if they did, then they would remain in $P(S')$ since $s_{r+1} < s_{r+2} < \cdots$.) But

$$s_{r+1} > \frac{1}{2}\sum_{j=k}^{r} s_j$$

and $\frac{1}{2}\sum_{j=k}^{r} s_j$ is the point of symmetry of $P((s_k,\cdots,s_r))$. Therefore,

$$\sum_{j=k}^{r} s_j - s_{r+1} < \frac{1}{2}\sum_{j=k}^{r} s_j$$

and by symmetry no gaps of length greater than w occur in $P((s_k, \cdots, s_r))$ between

$$\sum_{j=k}^{r} s_j - s_{r+1} \text{ and } \sum_{j=k}^{r} s_j - s_h .$$

Thus, no gaps of length greater than w occur between s_h and

$$\sum_{j=k}^{r} s_j - s_h = s_{r+2} - s_{k+1} - \epsilon_h + \epsilon_{k+1} - s_h$$

provided that r is sufficiently large. Now consider $P((s_k, \cdots, s_{r+3}))$. Since

$$s_{r+1} + s_{r+2} = s_{r+3} + (-1)^{r+1}$$

then $s_{r+1} + s_{r+2} + p$ and $s_{r+3} + p$ are elements of $P((s_k, \cdots, s_{r+3}))$ which differ by 1 whenever p is an element of $P((s_k, \cdots, s_r))$. Hence, since in $P((s_k, \cdots, s_r))$ there are no gaps of length greater than w between s_h and $\sum_{j=k}^{r} s_j - s_h$, then in $P((s_k, \cdots, s_{r+3}))$ there are no gaps of greater length than w - 1 between

$$s_h + s_{r+3} \text{ and } \sum_{j=k}^{r} s_j - s_h + s_{r+3} .$$

Similarly, consider $P((s_k, \cdots, s_{r+4}))$. Since

$$s_{r+2} + s_{r+3} = s_{r+4} + (-1)^{r+2}$$

and there are no gaps in $P((s_k, \cdots, s_{r+1}))$ of length greater than w between s_h and $\sum_{j=k}^{r+1} s_j - s_h$, then there are no gaps in $P((s_k, \cdots, s_{r+4}))$ of length greater than w - 1 between

$$s_h + s_{r+4} \text{ and } \sum_{j=k}^{r+1} s_j - s_h + s_{r+4} .$$

In general, for $q > 0$ since $s_{r+q} + s_{r+q+1} = s_{r+q+2} + (-1)^{r+q}$ and there are no gaps in $P((s_k, \cdots, s_{r+q-1}))$ of length greater than w between s_h and $\sum_{j=k}^{r+q-1} s_j - s_h$, then there are no gaps in $P((s_k, \cdots, s_{r+q+2}))$ of length greater than $w - 1$ between $s_h + s_{r+q+2}$ and $\sum_{j=k}^{r+q-1} s_j - s_h + s_{r+q+2}$. But

$$\begin{aligned}\sum_{j=k}^{r+q-1} s_j - s_h + s_{r+q-2} &= s_{r+q+1} - s_{k+1} - \epsilon_{r+q+1} + \epsilon_{k+1} - s_h + s_{r+q+2} \\ &= s_{r+q+3} + (-1)^{r+q+1} - s_{k+1} - s_h - \epsilon_{r+q+1} + \epsilon_{k+1} \\ &\geq s_{r+q+3} - s_{k+1} - s_h - 2 \ .\end{aligned}$$

Therefore, if we let

$$r' = s_{k+1} + s_h + 2$$

then for all sufficiently large z, there are no gaps in $P((s_k, \cdots, s_z))$ of length greater than $w - 1$ between $s_z + r'$ and $s_{z+1} - r'$ (since the preceding argument is valid for $q > 0$ and all sufficiently large r). This completes the proof of (b) and the theorem.

3. CONCLUDING REMARKS

Examples of sequences of positive integers which satisfy both (C) and (D) are rather elusive. It would be interesting to know if there exists such a sequence, say $T = (t_1, t_2, \cdots)$, which is essentially different from S, e.g., such that

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} \neq \frac{1 + \sqrt{5}}{2} \ .$$

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REFERENCE

1. J. L. Brown, "On Complete Sequences of Integers," Amer. Math. Monthly, 68 (1961) pp. 557–560.