ON NORRIE'S IDENTITY

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The first example expressing a biquadrate as the sum of four biquadrates was given by Norrie (University of St. Andrews 500th Anniversary Memorial vol., Edinburgh, 1911, 89). I give a simple demonstration of this result:

$$442^2 - 272^2 = 170\cdot 714 = 17^2\cdot 420,$$

hence $442^2 - 3\cdot 17^2 = 272^2 + 289\cdot 417 = 272^2 + 353^2 - 64^2$, but $3\cdot 17 = 2\cdot 26 - 1$, so

$$442^2 - 2\cdot 26\cdot 17 + 17 = 442^2 - 2\cdot 442 + 17 = 441^2 + 4^2 = 21^4 + 2^4 = 272^2 + 353^2 - 8^4.$$

Hence, $353^2 + 272^2 = 2^4 + 8^4 + 21^4$, but $353^2 - 272^2 = 81\cdot 625 = 15^4$, so $353^4 = 30^4 + 120^4 + 272^4 + 315^4$.

A FIBONACCI-LIKE SEQUENCE OF COMPOSITE NUMBERS

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Introduction. Let $S(L_0, L_1) = (L_0, L_1, L_2, \cdots)$ be a sequence of integers which satisfy the recurrence

$$L_{n+2} = L_{n+1} + L_n, \qquad n = 0, 1, 2, \cdots.$$

It is clear that the values of L_0 and L_1 determine $S(L_0, L_1)$, e.g., $S(0, 1)$ is just the sequence of Fibonacci numbers. It is not known whether or not infinitely many primes occur in $S(0, 1)$. On the other hand, if there is a prime p which divides both L_0 and L_1, then all the terms of $S(L_0, L_1)$ are divisible by p and in this case it is easily shown that only a finite number of the L_n can be prime. In this paper we exhibit two integers M and N with the following properties:

1. M and N are relatively prime.
2. No term of $S(M, N)$ is a prime number.

Preliminary remarks. Let L_0 and L_1 be arbitrary integers. Denote the nth Fibonacci number by F_n (where F_n is defined for *all* integers n by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, i.e., $F_{-1} = 1$, $F_{-2} = -1$, etc.).

For any $m \geqq 0$ we have

$$L_m = 1\cdot L_m + 0\cdot L_{m+1} = F_{-1}L_m + F_0 L_{m+1}$$
$$L_{m+1} = 0\cdot L_m + 1\cdot L_{m+1} = F_0 L_m + F_1 L_{m+1}$$
$$L_{m+2} = 1\cdot L_m + 1\cdot L_{m+1} = F_1 L_m + F_2 L_{m+1}.$$

Since $(F_n L_m + F_{n+1} L_{m+1}) + (F_{n+1} L_m + F_{n+2} L_{m+1}) = (F_{n+2} L_m + F_{n+3} L_{m+1})$, then by induction on n, it follows that

$$L_{m+n} = F_{n-1} L_m + F_n L_{m+1} \tag{1}$$

for any $m, n=0, 1, 2, \cdots$. Let p be a prime and let $r(p)$ denote the rank of apparition of p in the F_n, i.e., $r(p)$ is the smallest positive integer r for which $F_r \equiv 0 \pmod{p}$. Then for any $m \geqq 0$, we have by (1)

$$L_{m+r(p)} = F_{r(p)-1}L_m + F_{r(p)}L_{m+1} \equiv F_{r(p)-1}L_m \pmod{p}.$$

Thus,

$$L_m \equiv 0 \Rightarrow L_{m+r(p)} \equiv 0 \Rightarrow L_{m+kr(p)} \equiv 0 \pmod{p}$$

for any integer $k \geqq 0$.

Construction of M and N. Consider the following table:

n	a_n	$r(a_n)$	b_n	n	a_n	$r(a_n)$	b_n
1	2	3	2	10	41	20	10
2	3	4	1	11	53	27	16
3	5	5	1	12	109	27	7
4	7	8	3	13	31	30	24
5	17	9	4	14	2207	32	15
6	11	10	2	15	5779	54	52
7	61	15	3	16	2521	60	60
8	47	16	7	17	1087	64	31
9	19	18	10	18	4481	64	63

The a_n are prime numbers and the corresponding ranks of apparition $r(a_n)$ are easily verified. It is now asserted that every integer belongs to at least one of the arithmetic progressions

$$A_n = \{r(a_n)k + b_n : k = 0, \pm 1, \pm 2, \cdots\}$$

for $n=1, 2, \cdots, 18$ (i.e., the A_n form a covering set for the integers). First we see that A_{17}, A_{18}, A_{14}, A_8, A_4, and A_2 cover all the odd integers. Then A_{15}, A_{12}, A_{11}, A_9, A_5, and A_1 cover the remaining integers except those of the form $6n$. Next, A_{13}, A_7, A_6, and A_3 represent all the remaining integers except those of the form $30n$. Finally, since A_{10} and A_{16} cover the integers of the form $30n$, then *all* integers belong to at least one A_i.

Now, notice that if $L_{b_n} \equiv 0 \pmod{a_n}$ then $L_{b_n+kr(a_n)} \equiv 0 \pmod{a_n}$ for $k=0, 1, 2, \cdots$, i.e., $L_x \equiv 0 \pmod{a_n}$ for any x in A_n. Thus, if L_0 and L_1 can be chosen so that $L_{b_n} \equiv 0 \pmod{a_n}$ for $n=1, 2, \cdots, 18$ then *every* term of $S(L_0, L_1)$ is divisible by some a_n. This choice is easily made, for if we take

$$(2) \qquad \begin{aligned} L_0 &\equiv F_{r(a_n)-b_n} \pmod{a_n} \\ L_1 &\equiv F_{r(a_n)-b_n+1} \pmod{a_n} \end{aligned}$$

then from the definition of L_m we have $L_m \equiv F_{r(a_n)-b_n+m} \pmod{a_n}$ for $m=0, 1, 2, \cdots$, and consequently,

$$L_{b_n} \equiv F_{r(a_n)-b_n+b_n} \equiv F_{r(a_n)} \equiv 0 \pmod{a_n}.$$

(Since the a_j are relatively prime in pairs, then the Chinese remainder theorem guarantees the existence of L_0 and L_1 satisfying (2) simultaneously for $n=1, 2, \cdots, 18$.)

I am very grateful to Mr. John Brillhart for his assistance in obtaining an explicit solution to (2). In particular, the smallest positive solution to (2) is given by

$$M = L_0 = 1786772701928802632268715130455793,$$

$$N = L_1 = 1059683225053915111058165141686995.$$

From the way in which M and N were constructed, it follows at once that all the terms of $S(M, N)$ are composite while a routine application of the euclidean algorithm shows that $(M, N) = 1$.

A GEOMETRICAL COINCIDENCE

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CD is a chord of circle (O) perpendicular to the hypotenuse AB of the inscribed right triangle ABC. The rest of the figure is self-explanatory.

THEOREM. *The diameter of (P) is equal to the sum of the radii of (Q) and (R).*

Proof. (a) Diameter of $(P) = CU + CV = (AC - AU) + (BC - BV) = AC + BC - AB$. (b) Let $AB = 2r$; $AE = 2r_1$; $EB = 2r_2$. Then $OE = r_1 - r_2$. Since $OQ^2 - ON^2 = NQ^2$, we have

$$(r_1 + r_2 - NQ)^2 - (r_1 - r_2 + NQ)^2 = NQ^2,$$

from which we find

$$NQ = 2\sqrt{(rr_1)} - 2r_1 = AC - AE.$$

Similarly, $RM = CB - EB$. So $RM + NQ = AC + CB - (AE + EB) = AC + CB - AB$.

Note. If CD passes through O, the three inscribed circles are equal.