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On Partitions of a Finite Set

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Abstract

A pair of partitions π_1, π_2, of a finite set S into disjoint non-empty subsets will be called *conjugate* if for each $s \in S$, the ordered pair $(\nu_1(s), \nu_2(s))$ determines s, where $\nu_i(s)$ denotes the cardinality of the subset of π_i to which s belongs. In this note we show that S has a pair of conjugate partitions if and only if the cardinality of S is not equal to 2, 5, or 9. Partitions of this type provide a short solution to a problem arising in circuit theory.

Introduction

Suppose we have a cable consisting of n indistinguishable wires with terminals at two points A and B, and suppose for each terminal at A it is desired to identify its mate at B. We shall assume that the only operations available for making such an identification are interconnecting sets of terminals at one end and testing for current flow in the terminals at the other end. For example, if all terminals at A are connected together, then a current can flow between any two terminals at B. Without this assumption, the desired identification would present no problem, since if we denote the terminals at A by A_i, $1 \leq i \leq n$ (and similarly for B), then we simply test to see if a current can flow between A_1 and B_1, A_1 and $B_2, \ldots$, until we find a B_{i_1} such that a current can flow between A_1 and B_{i_1} and consequently we know A_i and B_{i_1} represent the same wire. We then use the same procedure on A_2, etc. For long cables, we shall restrict ourselves further to procedures of the following type:

Certain connections are made at A. We then go to B and make tests and, using the test results, certain connections. We finally come back to A, disconnect the connections initially made, and perform further tests. The information now in hand should be enought to determine for each j the terminal pairs A_j and B_{i_j} of wire j.

The following ingenious algorithum for solving this problem is due to my colleague K. C. Knowlton. Before presenting the general solution, a typical specific example will be given. Consider the case $n = 6$. Define the partitions P: $\{1, 2, 3\}$, $\{4, 5\}$, $\{6\}$, and P': $\{1, 4, 6\}$, $\{2, 5\}$, $\{3\}$, of the integers $\{1, 2, 3, 4, 5, 6\}$. If we let $f(i)$ denote the number of elements in the set of P which contains i and $f'(i)$ the similar function for the partition P', then we have the table

i	$(f(i), f'(i))$
1	(3,3)
2	(3,2)
3	(3,1)
4	(2,3)
5	(2,2)
6	(1,3)

We note that in the table all the ordered pairs $(f(i), f'(i))$ are *distinct*. To identify the terminals in our cable of six wires we first label those at A by $A_1, A_2, \ldots, A_6$ and connect them as shown in Figure 1(a). We now test at B to decide which wires have been connected at A.

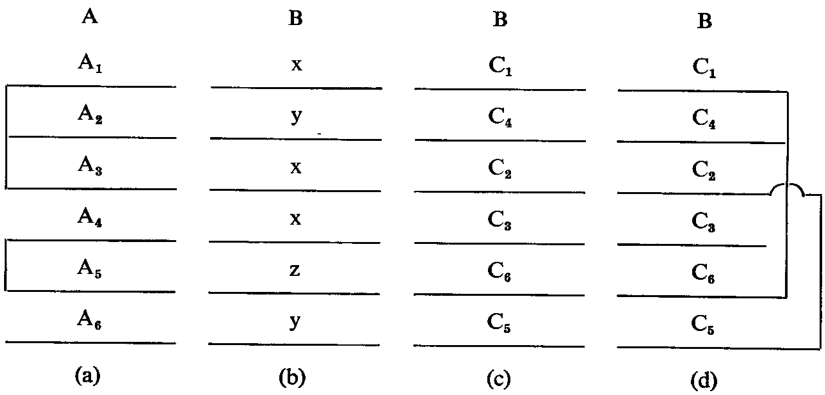

FIGURE 1

Suppose, for example, we find that we have the situation indicated in Figure 1(b) where two wires with the same symbol are joined at A. We label the x's with C_1, C_2 and C_3 (arbitrarily), the y's with C_4 and C_5 and the z with C_6, say, as in Figure 1(c). We next connect the terminals at B together according to the partition P', i.e., as in Figure 1(d). Finally we go back to A, disconnect the connections initially made there, and test to decide how many wires a given wire at A has been joined to at B. For example, suppose we find that wire A_1 is now connected to exactly one other wire at B. Since we know that initially A_1 was in a set of *three* wires which were connected at A and now it is in a set of *two* wires which are connected at B and, since the pair of $(f(i), f'(i)) = (3,2)$ occurs only for $i = 2$, then we can conclude that A_1 must correspond to C_2. Similarly if it happens that A_5 is now connected to two other wires at B (so that A_5 belongs to a set of *three* wires which have been joined) then A_5 must correspond to C_4, etc.

The general solution may be described in the following way. Let I_n denote the set of integers $\{1, 2, \ldots, n\}$. *Suppose* there exists a pair of partitions of I_n, say $P: P_1, P_2, \ldots, P_k$ and $P': P_1', P_2', \ldots, P_{k'}'$, such that, if $f(j)$ demotes the cardinality of the subsets P_i which contains j (with $f'(j)$ defined similarly), then the map $j \to (f(j), f'(j))$ is a $1-1$ mapping of I_n into $I_n \times I_n$. We shall call such a pair of partitions *conjugate*. For $1 \leq j \leq k$, we first connect all wires together at A which have subscripts that belong to the same P_j. We then go to B and, by suitable testing, determine the subsets $S_1, \ldots, S_k$ of the B_i which have been connected together at A. We next relabel the B_i by $C_1, C_2, \ldots, C_n$ so that for any S_r the set of indices of the C_i which occur in S_r is exactly one of the P_j. Now we connect the C_i together to form subsets T_j, $1 \leq j \leq k'$, in such a way that, for any j, the set of indices of the C_i which are in T_j is just P_j'. Finally we go back to A, disconnect all connections previously made there, and by suitable testing decide which A_i have been connected at B. We are now in a position to determine which labels represent the same wire. For if we take any wire, say A_u, we know that at B it belongs to a T_j which has, say, p elements (where we can determine p). Since we also know the cardinality of the S_i to which A_u belongs, say q, then, by the way the S_i and T_i were constructed and by the hypothesis that all the pairs $(p, q) = (f(u), f'(u))$ are distinct, we can determine the unique C_{i_u} and hence the B_{j_u} such that A_u and B_{j_u} represent the same wire.

It is the purpose of this note to prove that pairs of conjugate partitions exist for I_n if and only if $n \neq 2, 5$, or 9. We also give a simple construction of a pair of conjugate partitions for I_n for each $n \neq 2, 5$, or 9.

Some Necessary Conditions

As usual let $|A|$ denote the cardinality of the set A. If $P: P_1, P_2, \dots, P_k$ is a partition of I_n let $C(P)$ denote the set $\{|P_1|, |P_2|, \dots, |P_k|\}$ where we shall assume from now on that $|P_1| \leq |P_2| \leq \cdots \leq |P_k|$.

Lemma 1. *If P and P' are conjugate partitions of I_n then $|P_k| = |P'_{k'}|$.*

Proof. Suppose $|P_k| = m$. Since P and P' are conjugate then $C(P')$ must contain at least m distinct elements. Consequently $|P'_{k'}| \geq m$ (since we assume that $|P_1'| \leq \cdots \leq |P'_{k'}|$). Applying the same argument to P' we see that $|P_k| \leq |P'_{k'}| \leq |P_k|$ and the lemma follows.

If P is a partition of I_n for which there exists a partition P' of I_n such that P and P' are conjugate then we shall call P *admissible*.

Lemma 2. *If P is admissible and $|P_k| = m$ then $C(p) = \{1, 2, \dots, m\}$.*

Proof. Suppose there exists j such that $1 \leq j \leq m$ and $j \notin C(P)$. Since $|P'_{k'}| = m$ by Lemma 1 then we must have $|C(P)| \geq m$ (since P is admissible). Therefore $|P_k| \geq m + 1$, which is a contradiction.

Lemma 3. *Suppose P is admissible, $|P_k| = m$ and n_j denotes the number of P_i such that $|P_i| = j$. Then $n_j \leq [m/j]$ (where $[x]$ denotes the greatest integer not exceeding x).*

Proof. Suppose there exists j such that $n_j > [m/j]$ and let P' be a partition of I_n which is conjugate to P. Since $r \in P_a$, $s \in P_b$ and $|P_a| = |P_b| = j$ imply that $f'(r) \neq f'(s)$ then we must have $|C(P')| \geq j \cdot n_j$. But by Lemmas 1 and 2 (since P' is also admissible)

$$m = |C(P)| = |C(P')| \geq j \cdot n_j \geq j([m/j] + 1) > j \cdot m/j = m$$

which is a contradiction. This proves the lemma.

We combine these lemmas to obtain

Theorem 1. *If P is an admissible partition of I_n and $|P_k| = m$ then we have*

$$\frac{m(m+1)}{2} \leq n \leq \sum_{j=1}^{m} j \left[\frac{m}{j}\right]. \tag{1}$$

PROOF. The left side of (1) follows from Lemma 2 since

$$C(p) = \{1, 2, \ldots, m\}$$

implies that

$$n = |I_n| = |P_1| + \cdots + |P_k| \geq 1 + 2 + \cdots + m = \frac{m(m+1)}{2}.$$

The remaining part of (1) follows from Lemma 3 since

$$n = \sum_{j=1}^{m} j \cdot n_j \leq \sum_{j=1}^{m} j \left[\frac{m}{j}\right]. \quad \text{Q.E.D.}$$

If we let $J(m)$ denote $\sum_{j=1}^{m} j[m/j]$ and $\Delta(m)$ denote $[m(m+1)]/2$ then we have

$$\Delta(1) = 1 = J(1), \qquad \Delta(2) = 3 < 4 = J(2), \qquad \Delta(3) = 6 < 8 = J(3)$$

and

$$\Delta(4) = 10 \leq \Delta(m) \quad \text{for} \quad m \geq 4.$$

This shows that no admissible partitions exist for I_2, I_5, or I_9.

SOME SUFFICIENT CONDITIONS

THEOREM 2. *If n satisfies $\Delta(m) \leq n \leq \Delta(m+1) - 2$ for some positive integer m then there exist conjugate partitions of I_n.*

PROOF. If we let r denote $n - \Delta(m)$, then $0 \leq r \leq m - 1$. Consider the following array:

	m	$m-1$	$\cdots$	3	2	1	1	$\cdots$	1
m	x	x	$\cdots$	x	x	x			
$m-1$	x	x	$\cdots$	x	*		x		
$\vdots$	$\vdots$	$\vdots$		x					
3	x	x							
2	x	x							
1	x								
1					x				
$\vdots$									
1									

A row (column) headed by j will be said to be "open" if there are less than j x's in that row (column). By "projecting" an x of the array, we shall mean deleting a particular x of the array, replacing it by * and placing a new x in the same row (column) and also in the first open column (row). Note that projecting an x increases the number of x's in the array by 1. The "coordinates" of an x in the array are denoted by the ordered pair (i, j) where i and j are the headings of the column and row, respectively, to which x belongs. In the diagram, the x with coordinates $(2, m-1)$ has been projected. The two new x's formed have coordinates $(1, m-1)$ and $(2,1)$.

We now start with the original triangular array of x's where we assume that the number of rows which have a heading of 1 is $r+1$ (and similarly for columns). We next project the x's which have coordinates $(j, m-j+1)$ for $2 \leq j \leq \min(r+1, m-1)$. If $r = m-1$ then in addition we place an x at the intersection of the last row and last column (so that this x has coordinates $(1, 1)$). The new array now has $\Delta(m) + r$ x's in it and it is not difficult to see that with this construction *each x has a unique pair of coordinates*. We form two partitions P and P' of I_n as follows: Replace the x's in the new array by the elements of I_n (so that every element of I_n is used) in an arbitrary fixed way. For each column (row) of the array, we form a subset $P_i(P_i')$ belonging to the partition $P(P')$ by letting $P_i(P_i')$ be the set of all the integers which belong to that row (column). It follows that $(f(j), f'(j))$ is just the pair of coordinates which j has in the array and consequently P and P' are conjugate partitions of I_n. This proves the theorem.

As an example of the above construction, let $n = 13 = \Delta(4) + 3$. We start with the array

	4	3	2	1	1	1	1
4	x	x	x	x			
3	x	x	x				
2	x	x					
1	x						
1							
1							
1							

and project two x's and add the extra x (since $r = 3 = m - 1$) to obtain

	4	3	2	1	1	1	1
4	x	x	x	x			
3	x	x	*		x		
2	x	*				x	
1	x						
1			x				
1		x					
1							x

We arbitrarily replace the x's by the elements of I_{13} to form

	4	3	2	1	1	1	1
4	1	2	3	4			
3	5	6	*		7		
2	8	*				9	
1	10						
1			11				
1		12					
1							13

from which we generate the conjugate partitions of I_{13}:

$$P : \{4\}, \{7\}, \{9\}, \{13\}, \{3, 11\}, \{2, 6, 12\}, \{1, 5, 8, 10\}$$
$$P': \{10\}, \{11\}, \{12\}, \{13\}, \{8, 9\}, \{5, 6, 7\}, \{1, 2, 3, 4\}.$$

The only n for which I_n has not been shown to have conjugate partitions are those of the form $\Delta(m) - 1$ and, indeed, we have already noted that no such partitions exist for I_2, I_5, or I_9. We fill this gap with

THEOREM 3. *If* $n = \Delta(m) - 1$ *for* $m \geq 4$ *then there exist conjugate partitions for* I_n.

PROOF. We start with the array

		$m-1$	$m-2$	$\cdots$	3	2	1	1	$\cdots$	1	2
							$\overbrace{\hphantom{1}}^{m-2}$				
	$m-1$	x	x	$\cdots$	x	x	x				
	$m-2$	x	x		x	x					
	$\vdots$	$\vdots$			x						
	3	x	x								
	2	x	x								
$m-2$ $\{$	1	x									
	1										
	$\vdots$										
	1										
	2										

and project the x's which have coordinates $(j, m-j)$ for $2 \leq j \leq m-2$. Next we project the x at $(3, m-2)$ and place an additional x at the intersection of the last row and column (so that it has coordinates $(2, 2)$). Note that the new points formed from the projection of the x at $(3, m-2)$ have coordinates $(2, m-2)$ and $(3, 2)$ which are distinct from the coordinates of any other x's in the array (since $m \geq 4$ and the x which was originally at $(2, m-2)$ has been projected). It is easily checked that all x's in the array have distinct coordinates so that by replacing the x's by the elements of I_n we can form conjugate partitions of I_n and the theorem is proved.

CONCLUDING REMARKS

It is interesting to note that, by extending the construction used in Theorem 3, it is possible to form an admissible partition P of I_n with $|P_k| = m$ and $n = J(m)$ thus achieving the upper bound derived in Theorem 1. However, it is not clear that if n is any integer such that $\Delta(m) \leq n \leq J(m)$ then there exists an admissible partition of I_n with $|P_k| = m$. Since Lemmas 2 and 3 show that for any admissible partition P with $|P_k| = m$, we must have $1 \leq n_j \leq [m/j]$ then it might be conjectured that any partition P with $1 \leq n_j \leq [m/j]$ is admissible. This is not the case, however, as the following example shows. Let

$n = 28$ and choose P so that $n_1 = 1$, $n_2 = 3$, $n_3 = 2$, $n_4 = 1$, $n_5 = 1$ and $n_6 = 1$. Suppose P' is a partition conjugate to P and let F denote $\{(f(j), f'(j)) \colon 1 \leq j \leq 28\}$. We shall derive a contradiction. Since the pair (1,6) must belong to F then $n_2' \leq 2$ and $n_3' = 1$. But (6,3), (3,3), and (2,3) $\in F$ so that neither (5,3) nor (4,3) belong to F. Hence (5,1), (5,2), (5,4), (5,5), and (5,6) $\in F$ and therefore, since (6,4), (5,4), (3,4), and (2,4) $\in F$ then (4,4) $\notin F$. But we must have (6,2), (5,2), (3,2), and (2,2) $\in F$ so that (4,2) $\notin F$. Thus we have shown that (4,4), (4,3), and (4,2) cannot belong to F which is impossible. Hence P is not an admissible partition of I_{28}.

It may be shown that, by using this procedure, the maximum number of circuit tests which must be made for the identification is essentially $2n \log_2 n$. On the other hand, information theory arguments show that the number of tests must be at least $\log_2 (n!) \approx n \log_2 n$.

It should be remarked that the function

$$J(m) = \sum_{j=1}^{m} j \left[\frac{m}{j} \right]$$

can easily be put into the equivalent form

$$J(m) = \sum_{j=1}^{m} \sum_{d \mid j} d.$$

In this form $J(m)$ is recognized as a well-studied number theoretic function about which statements such as

$$J(m) \sim \frac{\pi^2}{12} m^2 + O(m \log m)$$

can be made (cf. [1]).

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Reference

1. W. J. LEVEQUE, *Topics in Number Theory*, Addison-Wesley, Reading, Mass., 1961, p. 121.