MATHEMATICAL NOTES

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THE SOLUTION OF A CERTAIN RECURRENCE

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In 1954, P. Turán [3] gave a proof of the identity

$$\binom{n+p}{p}^2 = \sum_{k=0}^{p} \binom{p}{k}^2 \binom{n+2p-k}{2p}$$

which he said appeared without proof in a book of the Chinese mathematician Le-Jen Shoo from 1867. This is equivalent to

$$\binom{n}{p}^2 = \sum_{k=0}^{p} \binom{p}{k}^2 \binom{n+k}{2p}$$

or

$$(1) \qquad \binom{n}{m}^2 = q_{nm} = \sum_{k=0}^{m} q_{mk} \binom{n+k}{2m}.$$

In one of the many successors to Turán's paper T. S. Nandjundiah [2] noticed that the Shoo identity is an instance of the following expansion of a product of binomial coefficients, namely

$$(2) \qquad \binom{m}{p}\binom{n}{q} = \sum_{k=0} \binom{n-m+p}{p-k}\binom{m-n+q}{k}\binom{n+k}{p+q}$$

(the upper limit of the sum is supplied by the convention that $\binom{a}{b}$ is zero if $a<0$, $b<0$, or $a<b$). Let

$$r_{nm} = \frac{1}{n+1}\binom{n-m}{m}\binom{n+1}{m+1} = \frac{1}{m+1}\binom{n}{m}\binom{n-1}{m}.$$

These numbers appeared in a study of a telephone traffic system with inputs from two sources made by John P. Runyon and are known locally as Runyon numbers; cf. J. A. Morrison [1]. It follows from (2) that

$$(m+1)r_{nm} = \binom{n}{m}\binom{n-1}{m} = \sum_{k=0} \binom{m+1}{m-k}\binom{m-1}{k}\binom{n+k}{2m}$$

or

$$(3) \qquad r_{nm} = \sum_{k=0} \frac{1}{m+1}\binom{m+1}{k+1}\binom{m-1}{k}\binom{n+k}{2m} = \sum_{k=0} r_{mk}\binom{n+k}{2m},$$

a relation similar to (1). The natural question arising is: what is the general solution of

$$x_{nm} = \sum_{k=0}^{m} x_{mk}\binom{n+k}{2m}. \tag{4}$$

Since the recurrence (4) leaves x_{nn} undetermined, this is the same as asking for the coefficient $X_k(n, m)$ in

$$x_{nm} = \sum_{k=0}^{m} X_k(n, m)x_{kk}. \tag{4a}$$

The answer is given by the following

THEOREM. *If* $n=0, 1, 2, \cdots, m=0, 1, \cdots, n$, *and*

$$x_{nm} = \sum_{k=0}^{m} x_{mk}\binom{n+k}{2m}, \tag{4}$$

then

$$x_{nm} = \sum_{k=0}^{m} \frac{2k+1}{m+k+1}\binom{n+k}{m+k}\binom{n-1-k}{m-k}x_{kk}, \qquad \textit{for } m < n \tag{5}$$

with arbitrary x_{kk}.

For a proof of the theorem, notice first that when $x_{nm}=r_{nm}$, $x_{kk}=r_{kk}=\delta_{0k}$, with δ_{nm} the Kronecker delta; hence

$$X_0(n, m) = r_{nm} = \frac{1}{m+1}\binom{n}{m}\binom{n-1}{m}.$$

Next, suppose that

$$x_{nm} = \frac{2p+1}{m+p+1}\binom{n-1-p}{m-p}\binom{n+p}{m+p}, \qquad p = 1, 2, \cdots, m.$$

Then, by (2)

$$\begin{aligned} x_{nm} &= \sum_{k=0} \frac{2p+1}{m+p+1}\binom{m-1-p}{k-p}\binom{m+p+1}{k+p+1}\binom{n+k}{2m} \\ &= \sum_{k=0} \frac{2p+1}{k+p+1}\binom{m-1-p}{k-p}\binom{m+p}{k+p}\binom{n+k}{2m} \\ &= \sum_{k=0} x_{mk}\binom{n+k}{2m} \end{aligned}$$

while $x_{kk}=\delta_{pk}$; hence

$$X_p(n, m) = \frac{2p+1}{m+p+1}\binom{n-1-p}{m-p}\binom{n+p}{m+p}, \qquad p = 0, 1, \cdots, m$$

and the theorem is proved.

The theorem leads to binomial identities whenever a particular solution of (4) (for which $x_{kk} \neq \delta_{pk}$, $p=0, 1, \cdots, m$) is known. Thus in the first instance $x_{nm} = q_{nm}$ yields

$$\binom{n}{m}^2 = \sum_{k=0}^{m} \frac{2k+1}{m+k+1}\binom{n-1-k}{m-k}\binom{n+k}{m+k} = \sum_{k=0}^{m} X_k(n, m)$$

since $q_{nn} = 1$.

A direct proof of this identity is as follows. First

$$\begin{aligned}\sum_0^m \frac{2k+1}{m+k+1}\binom{n-1-k}{m-k}\binom{n+k}{m+k} &= \sum_0^m \frac{2k+1}{n-m}\binom{n-1-k}{m-k}\binom{n+k}{m+k+1} \\ &= \sum_0^m \frac{2m+1}{n-m}\binom{n-1-k}{m-k}\binom{n+k}{m+k+1} \\ &\quad - 2\sum_0^m \frac{m-k}{n-m}\binom{n-1-k}{m-k}\binom{n+k}{m+k+1} \\ &= f_{nm} - g_{nm}.\end{aligned}$$

Next we have

$$\begin{aligned} f_{nm} &= \frac{2m+1}{n-m}\sum_0^m \binom{n-m+k-1}{k}\binom{n+m-k}{2m+1-k} \\ &= \frac{2m+1}{n-m}\sum_{m+1}^{2m+1} \binom{n-m+k-1}{k}\binom{n+m-k}{2m+1-k} \\ &= \frac{2m+1}{2n-2m}\sum_0^{2m+1} \binom{n-m+k-1}{k}\binom{n+m-k}{2m+1-k} \\ &= \frac{2m+1}{2n-2m}\binom{2n}{2m+1} = \binom{2n}{2m} \end{aligned}$$

(the next to last step uses one form of the Vandermonde relation). Also

$$g_{nm} = 2\sum_1^m \binom{n-1-k}{m-1-k}\binom{n+k}{m+k+1} = 2\sum_0^{m-1}\binom{n-m+k}{k}\binom{n+m-1-k}{2m-k}$$

and

$$\binom{2n}{2m} = \sum_0^{2m} \binom{n-m+k}{k}\binom{n+m-1-k}{2m-k}$$
$$= \sum_0^{m-1} \binom{n-m+k}{k}\binom{n+m-1-k}{2m-k}$$
$$+ \sum_0^{m} \binom{n-m-1+k}{k}\binom{n+m-k}{2m-k}$$
$$= \tfrac{1}{2}\cdot q_{nm} = \sum_0^{m} \binom{n-m-1+k}{k}\left[\binom{n+m-k-1}{2m-k}\right.$$
$$\left. + \binom{n+m-k-2}{2m-k-1} + \cdots + \binom{n}{m+1} + \binom{n}{m}\right]$$
$$= g_{nm} + \binom{n}{m}^2$$

which proves the identity.

Notice that

$$(2m+1)^{-1}f_{nm} = (2m+1)^{-1}\binom{2n}{2m} = \sum_0^{m} \frac{1}{m+k+1}\binom{n-1-k}{m-k}\binom{n+k}{m+k}$$

which is equation (5) with $x_{kk} = (2k+1)^{-1}$; hence

$$x_{nm} = (2m+1)^{-1}\binom{2n}{2m}$$

is a solution of (4) and

$$\frac{1}{2m+1}\binom{2n}{2m} = \sum_0^{m} \frac{1}{2k+1}\binom{2m}{2k}\binom{n+k}{2m}$$

or

$$\binom{2n}{2m} = \sum_0^{m} \binom{2m+1}{2k+1}\binom{n+k}{2m}.$$

A further result, which we do not take space to prove, is

$$\binom{2n+1}{2m} = \sum_0^{m} \binom{2m+1}{2k}\binom{n+k}{2m}$$

which is the x_{nm} with $x_{kk} = 2k+1$. Since sums and differences of solutions of (4) are also solutions, it follows that

$$x_{nm} = \frac{1}{2}\left[\binom{2n+1}{2m} - \binom{n}{m}^2\right]$$

is the solution for which $x_{kk} = k$.

References

1. J. A. Morrison, A certain functional-difference equation, Duke Math. J., 31 (1964) 445–448.
2. T. S. Nandjundiah, Remark on a note of P. Turán, this MONTHLY, 65 (1958) 354.
3. P. Turán, On a problem in the history of Chinese mathematics, Mat. Lapok, 5 (1954) 1–6.

ON THE TOTIENT FUNCTIONS OF JORDAN AND ZSIGMONDY

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Introduction. K. Zsigmondy (see [2], p. 152) devised a function to determine the number of elements of a certain order in a finite abelian group.

In this note it will be shown that Zsigmondy's function can be described completely by use of Jordan's totient function (see [2], p. 147). The proof is elementary and is much simpler than the lengthy combinatorial proofs of the formula found in the literature (see, for example, [1]).

I. In order to translate the problem into number-theoretic concepts, we make the following definitions:

DEFINITION. *Let n and k be positive integers. A k-tuple $\{a_1, a_2, \cdots, a_k\}$ of positive integers is called a prime sequence for n (of length k) provided $1 \leqq a_i \leqq n$ and $(a_1, a_2, \cdots, a_k, n) = 1$ (the parentheses denote the greatest common divisor).*

DEFINITION. *If n and k are positive integers, then $J_k(n)$ denotes the number of distinct prime sequences for n, each of length k. $J_0(n)$ is defined to be zero.*

DEFINITION. *Let $m, n_1, n_2, \cdots, n_s$ be fixed positive integers. An s-tuple $\{a_1, a_2, \cdots, a_s\}$ of positive integers is called a primitive sequence for m (with respect to $n_1, \cdots, n_s$) provided*

(1) $1 \leqq a_i \leqq n_i$ $(i = 1, 2, \cdots, s)$ *and*

(2) *m is the smallest positive integer such that* $ma_i \equiv 0 \pmod{n_i}$ $(i = 1, 2, \cdots, s)$.

DEFINITION. *If m is a positive integer then $\psi(m) = \psi(m; n_1, n_2, \cdots, n_s)$ denotes the number of distinct primitive sequences for m (with respect to $n_1, n_2, \cdots, n_s$).*

Thus if G is a finite abelian group with independent generators $g_1, g_2, \cdots, g_s$ of order $n_1, n_2, \cdots, n_s$, respectively, then $\psi(m)$ is the number of elements of G of order m.

II. THEOREM. *ψ is a multiplicative function.*