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Ramsey's theorem for N-parameter sets: an outline

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INTRODUCTION

In 1930, F. P. Ramsey [7, 9] proved the following theorem:

Theorem [Ramsey]: Let ℓ, k, r be positive integers. Then there is a number $N_{\cdot} = N(\ell, k, r)$ depending only on ℓ, k and r with the following property: If S is a set with at least N elements, and if all the subsets of S with k elements are divided into r classes in any way, then there is some subset of ℓ elements with all of its subsets of k elements in a single class.

Since this theorem appeared there has been interest in finding generalizations, applications and analogues of it. The work presented here was motivated by a conjecture made by Gian-Carlo Rota, a geometric analogue to Ramsey's theorem, which can be stated as follows:

Conjecture [Rota]: Let ℓ, k, r be nonnegative integers, and F a field of q elements. Then there is a number $N = N(q, r, \ell, k)$ depending only on q, r, ℓ and k with the following property: If V is a vector space over F of dimension at least N, and if all the k-dimensional subspaces of V are divided into r classes in any way, then there is some ℓ-dimensional subspace with all of its k-dimensional subspaces in a single class.

This conjecture is obtained from the statement of Ramsey's theorem essentially by replacing the notions of set and cardinality by those of vector space and dimension, respectively, if we replace the notion of vector space with that of affine space, then we obtain another conjecture. This conjecture is actually equivalent to Rota's conjecture [2, 8].
In this paper we outline the proof of another analogue to Ramsey's theorem, in which we replace the notion of n-dimensional affine space by the notion of n-parameter set, which we define later. The n-parameter sets are similar to n-dimensional affine spaces in certain ways, and, in fact, by appropriate choice of certain variables we can obtain results for vector and affine spaces. In particular, the affine conjecture is shown to be true for the cases of $k = 0$ and $k = 1$, with any choice for ℓ, r and q. This implies that Rota's conjecture is true for $k = 1$ and $k = 2$ [2, 8]. Some other interesting results which follow from the n-parameter set analogue are presented as corollaries to the main result.

In general, we shall not present the details of the proofs of various assertions since they are rather long and will appear elsewhere. What we shall attempt to do instead is to give a rough indication of the proofs and to show how the results may be applied.

k-PARAMETER SETS

All of the aforementioned analogues to Ramsey's theorem are just statements about some special kinds of subsets of certain sets and their inclusion relationships.

Ramsey's theorem itself can be thought of thus as a statement about the lattices of subsets of finite sets; Rota's conjecture refers to the lattices of subspaces of finite vector spaces; the affine analogue concerns the partially-ordered sets of the subspaces of finite affine spaces. So also is the n-parameter set analogue a statement about partially ordered sets of special subsets of certain sets. We give here a precise definition of a k-parameter set followed by some (less formal) convenient notation with which the concepts involved can be more readily digested.

Let $A = \{a_1, a_2, \dots, a_t\}$ be a finite set with $t \geq 2$. Let $H : A \to A$ be a permutation group acting on A. For $a \in A$, $\sigma \in H$ the action is denoted by $a \to a^{\sigma}$. Also, for $\sigma_1, \sigma_2 \in H$, $\sigma_1 \cdot \sigma_2 \in H$ is defined by $a^{\sigma_1 \cdot \sigma_2} = (a^{\sigma_1})^{\sigma_2}$ for all $a \in A$. For a nonempty subset $B \subseteq A$, let $\bar{B} = \{\bar{b} : b \in B\}$ be the set of constant maps of A into A given by $x^{\bar{b}} = b$ for $x \in A$, $\bar{b} \in \bar{B}$. A^t denotes the cartesian product $A \times A \times \dots \times A$ (t factors) which is just $\{(x_1, \dots, x_t) : x_i \in A, \ 1 \leq i \leq t\}$.

For $x = (x_1, \dots, x_t) \in A^t$, $\sigma \in H$, we define an action of $H : A^t \to A^t$ by

$$x^{\sigma} = (x_1, \dots, x_t)^{\sigma} = (x_1^{\sigma}, \dots, x_t^{\sigma}) \in A^t .$$

Similarly $\bar{B}$ acts on A^t by

$$x^{\bar{b}} = (x_1, \dots, x_t)^{\bar{b}} = (x_1^{\bar{b}}, \dots, x_t^{\bar{b}}) = (b, \dots, b) \in A^t$$

for $x \in A^t$, $\bar{b} \in \bar{B}$.

For fixed integers $n > 0$ and $0 \leq k \leq n$, let $\Pi = \{S_0, S_1, \dots, S_k\}$ be a partition of the set $I_n = \{1, 2, \dots, n\}$ with $S_i \neq \emptyset$ for $1 \leq i \leq k$. $S_0 = \emptyset$ is possible. Let $f : I_n \to H \cup \bar{B}$ be a mapping with the property:

$$f(i) \in \bar{B} \quad \text{if} \quad i \in S_0$$

$$f(i) \in H \quad \text{if} \quad i \in I_n - S_0 .$$

The set $P = (A, \bar{B}, H, \Pi, f, n, k) = P$ is defined by

$$P = \bigcup_{1 \le i_0, \dots, i_k \le t} \left\{ (x_1, \dots, x_n) : x_j = a_{i_y}^{f(j)} \quad \text{if} \quad j \in S_y \right\} \subseteq A^n .$$

Definition: A subset $P \subseteq A^n$ is said to be a k-parameter set in A^n if $P = P(A, \bar{B}, H, \pi, f, n, k)$ for some meaningful choice of these variables. What this means is the following. Let us write π symbolically as:

$$\overbrace{\qquad\qquad}^{S_0}\ \overbrace{\qquad\qquad}^{S_1} \qquad\qquad \overbrace{\qquad\qquad}^{S_k}$$

We imagine that we have bunched together the elements in the blocks of the partition π. With each $i \in I_n$ we associate an element $f(i) \in \bar{B} \cup H$. We can write this as

$$[\overbrace{\bar{a} \ \dots \ \bar{b}}^{S_0} \ \overbrace{\pi_1 \ \cdots \ \delta_1}^{S_1} \quad \cdots \quad \overbrace{\pi_k \ \cdots \ \delta_k}^{S_k}]$$

where $\bar{a}, \dots, \bar{b} \in \bar{B}$, $\pi_1, \dots, \delta_k, \dots, \pi_k, \dots, \delta_k \in H$. With ℓ_0 defined by

$$\ell_0 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{pmatrix} \in A^t$$

(we occasionally write elements of A^t as column vectors when this is useful for our purposes), the preceding is shorthand notation for

$$\left[\overbrace{\ell_0^{\bar{a}} \ \dots \ \ell_0^{\bar{b}}}^{S_0} \quad \overbrace{\ell_0^{\pi_1} \ \dots \ \ell_0^{\delta_1}}^{S_1} \quad \dots \quad \overbrace{\ell_0^{\pi_k} \ \dots \ \ell_0^{\delta_k}}^{S_k} \right]$$

which we can write as

$$\begin{array}{c}
\begin{array}{cccccccccc}
\multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{S_0}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{S_1}} & & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{S_k}}
\end{array} \\
\begin{bmatrix}
a_1^{\bar{a}} & \cdots & a_1^{\bar{b}} & a_1^{\pi_1} & \cdots & a_1^{\delta_1} & \cdots & a_1^{\pi_k} & \cdots & a_1^{\delta_k} \\
a_2^{\bar{a}} & \cdots & a_2^{\bar{b}} & a_2^{\pi_1} & \cdots & a_2^{\delta_1} & \cdots & a_2^{\pi_k} & \cdots & a_2^{\delta_k} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_t^{\bar{a}} & \cdots & a_t^{b} & a_t^{\pi_1} & \cdots & a_t^{\delta_1} & \cdots & a_t^{\pi_k} & \cdots & a_t^{\delta_k}
\end{bmatrix}
\end{array}$$

which, of course, is just

$$\begin{array}{c}
\begin{array}{cccccccccc}
\multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{S_0}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{S_1}} & & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{S_k}}
\end{array} \\
\begin{bmatrix}
a & \cdots & b & a_1^{\pi_1} & \cdots & a_1^{\delta_1} & \cdots & a_1^{\pi_k} & \cdots & a_1^{\delta_k} \\
a & \cdots & b & a_2^{\pi_1} & \cdots & a_2^{\delta_1} & \cdots & a_2^{\pi_k} & \cdots & a_2^{\delta_k} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a & \cdots & b & a_t^{\pi_1} & \cdots & a_t^{\delta_1} & \cdots & a_t^{\pi_k} & \cdots & a_t^{\delta_k}
\end{bmatrix}
\end{array}$$

Now, consider an n-tuple $x = (x_1, \ldots, x_n) \in A^n$ formed in the following way:

$$x = \Big(\overbrace{a, \ldots, b}^{S_0},\ \overbrace{a_{i_1}^{\pi_1}, \ldots, a_{i_1}^{\delta_1}}^{S_1},\ \ldots,\ \overbrace{a_{i_k}^{\pi_k}, \ldots, a_{i_k}^{\delta_k}}^{S_k}\Big)$$

where $1 \le i_1, i_2, \ldots, i_k \le t$. We can think of x as being formed by taking "row cross-sections" under the various S_i and independently piecing these together. The set of all such x forms the set P. Since each $\pi_i, \ldots, \delta_i$ is a permutation of A, then a different choice of "row cross-sections" results in a different n-tuple x. Hence, $|P| = t^k$.

Thus, P is a k-parameter set in A^n iff P can be generated by some expression of the form

$$(1) \qquad [\,\overbrace{a \cdots b}^{S_0}\ \overbrace{\pi_1 \cdots \delta_1}^{S_1}\ \cdots\ \overbrace{\pi_k \cdots \delta_k}^{S_k}\,]$$

If P_ℓ is an ℓ-parameter set in A^n, we say that P_k is a k-parameter subset of P_ℓ if P_k is a k-paramter set in A^n and P_k is a subset of P_ℓ (with the same $A, \bar{B}, H, n$).

We point out here that a set of 2^k points of A^n may possibly have many representations of the form (1). It is a k-parameter set, however, iff there is at least one such representation.

For example, for any choice of $\sigma_1, \sigma_2, \ldots, \sigma_n \in H$ the set denoted by $[\,\overbrace{\sigma_1}^{S_1}\ \overbrace{\sigma_2}^{S_2} \ldots \overbrace{\sigma_n}^{S_n}\,]$ is just A^n, which is an n-parameter subset of itself.

We next state several facts about k-parameter sets whose proofs we omit.

(i) Let $P = P(A, \bar{B}, H, \Pi, f, n, k)$ be a k-parameter set in A^n and $\beta_i \in H$, $1 \le i \le k$. Define $f' : I_n \to H \cup \bar{B}$ by

$$f'(j) = \begin{cases} \beta_i f(j) & \text{for} \quad j \in S_i, \quad 1 \le i \le k, \\ f(j) & \text{for} \quad j \in S_0 \end{cases}$$

Then $P' = P(A, \bar{B}, H, \pi, f', n, k) = P$.

(ii) Let $P = P(A, \bar{B}, H, \pi; f, n, \ell)$ be an ℓ-parameter set in A^n. The general k-parameter subset $P_k \subseteq P_\ell$ is formed as follows: Choose an unrefinement π' of π, say $\pi' = \{S_0', S_1', \ldots, S_k'\}$ with $S_0 \subseteq S_0'$ and $S_i' \neq \phi$, $i > 0$. For each $S_i \subseteq S_0'$, $i > 0$, choose $\tau_i \in \bar{B}$; for each $S_i \nsubseteq S_0'$, choose $\tau_i \in H$.

Define $I_n \to H \cup \bar{B}$ by $f'(j) = \begin{cases} \tau_i f(j), & j \in S_i, \quad i > 0, \\ f(j), & j \in S_0. \end{cases}$

Then $P_k = P(A, \bar{B}, H, \pi', f', n, k)$ is a k-parameter set in A^n, $P_k \subseteq P_\ell$ and all k-parameter subsets of P_ℓ can be obtained this way.

CONSTRUCTION OF $*$-SETS

We now give a new construction which will be essential in the remainder of the paper. We retain the notation of the preceding section. Define

$$L_{\bar{A}} = \{ \ell_0^{\bar{a}} : a \in A \} = \{ (a, \ldots, a) : a \in A \} \subseteq A^t,$$

$$L_{\bar{B}} = \{ \ell_0^{\bar{b}} : b \in B \},$$

$$L_H = \{ \ell_0^{\sigma} : \sigma \in H \},$$

$$L = L_{\bar{A}} \cup L_H = \{ \ell_1, \ldots, \ell_u \} \subseteq A^t.$$

For $x = (x_1, \ldots, x_u) \in L^u$, $\sigma \in H$, we define an action of $H : L^u \longrightarrow L^u$ by

$$x^{\sigma} = (x_1^{\sigma}, \ldots, x_u^{\sigma}).$$

Similarly, define $\bar{B} : L^u \longrightarrow L^u$ by

$$x^{\bar{b}} = (x_1^{\bar{b}}, \ldots, x_u^{\bar{b}}).$$

For all $\ell, m \in L$, define the map $\bar{\ell} : L \longrightarrow L$ by

$$m^{\bar{\ell}} = \ell .$$

This induces a map $\bar{\ell} : L^u \longrightarrow L^u$ by

$$x^{\bar{\ell}} = (x_1^{\bar{\ell}}, \ldots, x_u^{\bar{\ell}}) = (\ell, \ldots, \ell) \in L^u .$$

Finally we make the following definitions:

$$\ell_0^* = (\ell_0^{\bar{a}_1}, \ldots, \ell_0^{\bar{a}_t}, \ell_0^{\sigma_1}, \ldots, \ell_0^{\sigma_h}) \in L^u ,$$

$$C = L_H \cup L_{\bar{B}} , \quad \bar{C} = \{ \bar{c} : c \in C \} = \overline{L_H} \cup \overline{L_{\bar{B}}} ,$$

$$L_H^* = \{ \ell_0^{*\sigma} : \sigma \in H \} , \quad L_{\bar{C}}^* = \{ \ell_0^{*\bar{c}} : \bar{c} \in \bar{C} \} .$$

As before, we have the notion of k-parameter sets in L^n.

For L^n we modify the notation slightly by writing a k-parameter set $P_k^* = P(L, \bar{C}, H, \Pi^*, g, n, k)$ as

$$\Big[\overbrace{\overbrace{\overline{\ell_0^{\bar{b}}} \cdots \overline{\ell_0^{\bar{d}}}}^{T_0^*} \quad \overbrace{\overline{\ell_0^{\pi_0}} \cdots \overline{\ell_0^{\delta_0}}}^{V_0^*}}^{S_0^*} \quad \overbrace{\pi_1 \cdots \delta_1}^{S_1^*} \quad \cdots \quad \overbrace{\pi_k \cdots \delta_k}^{S_k^*} \Big]$$

where $\overline{\ell_0^{\bar b}}, \dots, \overline{\ell_0^{\bar d}} \in \overline{L_{\bar B}}$ and $\ell_0^{\pi_0}, \dots, \ell_0^{\delta_0} \in L_H$ (i.e., $\pi_0, \dots, \delta_0 \in H$). Slightly expanded, this is

$$u\left\{\begin{array}{llll}
\overbrace{\overbrace{\ell_0^{\bar b}\cdots\ell_0^{\bar d}}^{T_0^*}\quad\overbrace{\ell_0^{\pi_0}\cdots\ell_0^{\delta_0}}^{V_0^*}}^{S_0^*} & \overbrace{\left(\ell_0^{\bar a_1}\right)^{\pi_1}\cdots\left(\ell_0^{\bar a_1}\right)^{\delta_1}}^{S_1^*} & \cdots & \overbrace{\left(\ell_0^{\bar a_1}\right)^{\pi_k}\cdots\left(\ell_0^{\bar a_1}\right)^{\delta_k}}^{S_k^*} \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots & \vdots \qquad\qquad \vdots & & \vdots \qquad\qquad \vdots \\
\ell_0^{\bar b}\cdots\ell_0^{\bar d}\quad\ell_0^{\pi_0}\cdots\ell_0^{\delta_0} & \left(\ell_0^{\sigma_h}\right)^{\pi_1}\cdots\left(\ell_0^{\sigma_h}\right)^{\delta_1} & \cdots & \left(\ell_0^{\sigma_h}\right)^{\pi_k}\cdots\left(\ell_0^{\sigma_h}\right)^{\delta_k}
\end{array}\right.$$

THE M MAP

We define a map $M : L^n \longrightarrow 2^{A^n}$ as follows: For $x = (x_1, \dots, x_n) \in L^n$, $x_i = (x_{i1}, \dots, x_{it}) \in L \subseteq A^t$, $1 \le i \le n$, let

$$M(x) = \left\{\begin{array}{l}
(x_{11}, x_{21}, \dots, x_{n1}), \\
(x_{12}, x_{22}, \dots, x_{n2}), \\
\quad\vdots \qquad \vdots \\
(x_{1t}, x_{2t}, \dots, x_{nt})
\end{array}\right\} \subseteq A^n$$

For $S \subseteq L^n$ we define $M(S)$ to be $\bigcup_{s \in S} M(S)$.

Suppose $P_\ell^* = P^*(L,\bar{C},H,\pi^*,g,n,\ell)$ is a ℓ-parameter set in L^n. It can be shown that if $V_0^* \neq \phi$ then $M(P_\ell^*)$ is just an $(\ell+1)$-parameter set $P_{\ell+1}$ in A^n. Furthermore if P_{k+1} is any $(k+1)$-parameter subset of $P_{\ell+1}$ in which T_0^* and V_0^* are not in the same block of the partition for P_{k+1} then there exists a k-parameter set P_k^* in L^n for which the following diagram is commutative:

$$\begin{tikzcd}
\exists\, P_k^* \arrow[r, phantom, "\subseteq"] \arrow[d, "M"'] & P_\ell^* \arrow[d, "M"] \\
P_{k+1} \arrow[r, phantom, "\subseteq"] & P_{\ell+1}
\end{tikzcd}$$

Before proceeding to the outline of the proof we make a remark on terminology. By an r-coloring of a set X we just mean a partition of X into r disjoint (possibly empty) classes. Of course, the " r colors" correspond to the r classes into which X is partitioned. In general, we shall use this "chromatic" terminology in preference to that of partitions and classes.

THE MAIN RESULT

Theorem: Given A,B,H and integers $k,r,t_1,\cdots,t_r$, there exists an $N = N(A,\bar{B},H,k,r,t_1,\ldots,t_r)$ such that if $n \geq N$ and $P_n = P(A,\bar{B},H,\pi,f,w,n)$ is any fixed n-parameter set in A^w, then for any r-coloring of the k-parameter subsets of P_n there is an i, $1 \leq i \leq r$, such that some t_i-parameter subset of P_n has all its k-parameter subsets the i^{th} color.

Proof: The proof will proceed basically by double induction on k and $t_1 + \ldots + t_k$. The proof for $k = 0$ and all t_i is relatively straightforward once certain notational difficulties have been overcome. For a fixed integer $k \geq 0$ assume the theorem has been established for this k and all values of $r,t_1,\cdots,t_r$. We prove the theorem for $k+1$. Of course, the theorem is immediate for $r = 1$, and it is true vacuously for

$t_1 + \cdots + t_n \leq (k+1)r - 1$ (since in this case for some i, $t_i < k+1$). Henceforth we assume that $r \geq 2$, $t_i \geq k+1$, and that for some $p \geq (k+1)r-1$ the theorem holds for all $t_1 + \ldots + t_r \leq p$ We must prove the theorem with the additional assumption, which we now make, that $t_1 + \cdots + t_r = p+1$.

Definition: Let $P_m = P(X, \bar{Y}, G, \Pi, f, w, m)$ be an m-parameter set in X^w, where Π is the partition $\{S_0, S_1, \cdots, S_m\}$. Then for $k \leq m$ and $1 \leq i \leq m$, an S_i-crossing k-parameter subset of P_m is a k-parameter subset $P_k = P(X, \bar{Y}, G, \Pi', f', w, k)$ with the partition $\Pi' = \{S'_0, S'_1, \ldots, S'_k\}$, and $S_i \nsubseteq S'_0$.

Now let L, C and the map M be as before. We state a lemma whose proof we omit.

Lemma 1: Let $P_{m+1} = P(A, \bar{B}, H, \Pi, f, w, m+1)$ be an $(m+1)$-parameter set in A^w with partition $\Pi = \{S_0, S_1, \cdots, S_{m+1}\}$. Let $\ell \geq 0$ be an integer. If $m \geq N(L, C, H, k, r, \underbrace{\ell, \ldots, \ell}_{r})$ (which is meanigful by the induction hypothesis), then for any fixed i, $1 \leq i \leq t$, and for any r-coloring of the $(k+1)$-parameter subsets of P_{m+1}, there is an S_i-crossing $(\ell+1)$-parameter subset $P_{\ell+1}$ of P_{m+1} such that for some j, $1 \leq j \leq r$, all the S_i-crossing $(k+1)$-parameter subsets of $P_{\ell+1}$ have the j^{th} color.

Let A, B, C, H, L and M be as before. Let $P^*_m = P(L, \bar{C}, H, \Pi^*, f, w, m)$ be an m-parameter set in L^w with partition $\Pi^* = \{V^*_0 \cup T^*_0 = S^*_0, S^*_1, \ldots, S^*_m\}$, $V^*_0 \neq \phi$. Then $P_{m+1} = M(P^*_m)$ is an $(m+1)$-parameter set in A^w. Let $P_{\ell+2}$ be a V^*_0-crossing $(\ell+2)$-parameter subset of P_{m+1},

$$P_{\ell+2} = [\overbrace{\bar{b} \ldots \bar{d}}^{S_0}\ \overbrace{\overbrace{\pi_0 \cdots \delta_0}^{V^*_0}\ \cdots\ \overbrace{\sigma_j \pi_j \cdots \sigma_j \delta_j}^{S^*_j}}^{S_1}\ \overbrace{\pi \cdots}^{S_2}\ \cdots\ \overbrace{\cdots \delta}^{S_{\ell+2}}].$$

Then $P_{\ell+2}$ is the disjoint union of t $(\ell+1)$-parameter subsets $P^{i}_{\ell+1}$, $1 \leq i \leq t$, none of which are V_0^*-crossing subsets, defined by

$$P^{i}_{\ell+1} = \left[\overbrace{\overbrace{\bar{b}\ldots\bar{d}}^{S_0}\ \ \overbrace{\overbrace{\overline{a_i^{\pi_0}}\ldots\overline{a_i^{\delta_0}}}^{V_0^*}\ \ \ldots\ \ \overbrace{\overline{a_i^{\sigma_j\pi_j}}\ldots\overline{a_i^{\sigma_j\delta_j}}}^{S_j^*}}^{S_1}}^{S_0'}\ \ \overbrace{\pi\ldots}^{S_2}\ \ \ldots\ \ \overbrace{\ldots\delta}^{S_{\ell+2}} \right].$$

Definition: The $P^{i}_{\ell+1}$ are called V_0^*-translates of each other in $P_{\ell+2}$ (or just translates when no confusion arises).

Remark 1: Let $P_{\ell+2}$ be a V_0^*-crossing $(\ell+2)$-parameter subset of P_{m+1}, as above, with V_0^*-translates $P^{i}_{\ell+1}$, and let P_{k+2} be a V_0^*-crossing $(k+2)$-parameter subset of $P_{\ell+2}$. Then

$$P_{k+2} = [\overbrace{\overbrace{\bar{b}\ldots\bar{d}}^{S_0}\ \ldots\ \overbrace{\overbrace{\pi_0\ldots\delta_0}^{V_0^*}\ \ldots\ \overbrace{\sigma_j\pi_j\ldots\sigma_j\delta_j}^{S_j^*}}^{S_1}\ \ldots\ \overbrace{\tau\ldots\eta}^{S_e}}^{S_1'}\ \overbrace{\pi'\ldots}^{S_2'}\ \ldots\ \overbrace{\ldots\delta'}^{S_{k+2}'}].$$

P_{k+2} is the disjoint union of the t V_0^*-translates P^{i}_{k+1}, where

$$P^i_{k+1} = \left[\overbrace{\overbrace{\overbrace{\bar{b}\cdots\bar{d}}^{S_0}}^{S_0} \cdots \overbrace{\overbrace{\overbrace{\overline{a_i^{\pi_0}}\cdots\overline{a_i^{\delta_0}}}^{V_0^*} \cdots \overbrace{\overline{a_i^{\sigma_j\pi_j}}\cdots\overline{a_i^{\sigma_j\delta_j}}}^{S_j^*}}^{S_1} \cdots \overbrace{\overline{a_i^{\tau}}\cdots\overline{a_i^{\eta}}}^{S_e}}^{S'_1}}^{S''_0} \overbrace{\pi'\cdots}^{S'_2} \cdots \overbrace{\cdots\delta'}^{S'_{k+2}} \right]$$

We see that $P^i_{k+1} = P^i_{\ell+1} \cap P_{k+2}$ because $P^i_{k+1} \subseteq P^i_{\ell+1}$ and $P_{k+2} \subseteq P_{\ell+2}$, and on the other hand, any point in $P^i_{\ell+1}$ and P_{k+2} must be in P^i_{k+1} as can be checked by verifying the inclusion properties of parameter sets.

Remark 2: If P_{k+1} is any $(k+1)$-parameter subset of $P^i_{\ell+1}$, then there is some V_0^*-crossing $(k+2)$-parameter subset of $P_{\ell+2}$ with $P_{k+2} = \bigcup_{j=1}^{t} P^j_{k+1}$, the P^j_{k+1} being V_0^*-translates, such that $P_{k+1} = P^i_{k+1}$. In particular, taking $P^i_{\ell+1}$ to be as in the definition, $P_{k+1} \subseteq P^i_{\ell+1}$ must look like

$$P_{k+1} = \left[\overbrace{\overbrace{\bar{b}\cdots\bar{d}}^{S_0} \cdots \overbrace{\overbrace{\overline{a_i^{\pi_0}}\cdots\overline{a_i^{\delta_0}}}^{V_0^*} \cdots \overbrace{\overline{a_i^{\sigma_j\pi_j}}\cdots\overline{a_i^{\sigma_j\delta_j}}}^{S_j^*}}^{S_1} \cdots \overbrace{\overline{a_\chi^{\tau}}\cdots\overline{a_\chi^{\eta}}}^{S_g}}^{S''_0} \overbrace{\pi'\cdots}^{S'_2} \cdots \overbrace{\cdots\delta'}^{S'_{k+2}} \right]$$

Then we can take

$$P_{k+2} = \Big[\overbrace{\overbrace{\bar{b} \dots \bar{d}}^{S_0} \;\dots\; \overbrace{\overline{a_x^{\tau}} \dots \overline{a_x^{\eta}}}^{S_g}}^{S'_0} \; \overbrace{\overbrace{\pi_0 \dots \delta_0}^{V_0^*} \;\dots\; \overbrace{\sigma_j \pi_j \dots \sigma_j \delta_j}^{S_j^*}}^{S'_1 = S_1} \; \overbrace{\pi' \dots}^{S'_2} \;\dots\; \overbrace{\dots \delta'}^{S'_{k+2}} \Big]$$

This choice of P_{k+2} is well-defined. That is, $S'_1 = S_1$ is the smallest set we can choose from S''_0 to generate a $(k+2)$-parameter set which is V_0^*-crossing and is contained in $P_{\ell+2}$ (since any such S'_1 must contain S_1). We shall refer to this particular P_{k+2} as the V_0^*-expansion of P_{k+1} in $P_{\ell+2}$.

Remark 3: It should be noted that if P_{k+1} is any $(k+1)$-parameter subset of $P_{\ell+2}$, then either P_{k+1} is a V_0^*-crossing $(k+1)$-parameter set, or $P_{k+1} \subseteq P_{\ell+1}^{(i)}$ for some i.

This follows from the way in which the $(k+1)$-parameter subsets of $P_{\ell+2}$ must be formed.

Definition: Let A, B, H be as above. Let P_{m+v} be an $(m+v)$-parameter subset of A^w with partition $\{S_0, S_1, \dots, S_v, V_1, \dots, V_m)$. For each $i = 1, 2, \dots, m$, P_{m+v} is the union of t disjoint $(m+v-1)$-parameter subsets $P^j_{(m+v-1),i}$, $1 \le j \le t$, which are V_i-translates of each other. Let P_{k+1} be a $(k+1)$-parameter subset of P_{m+v} which is V_i-crossing for at least one i. Let $\ell = m - \max\{i : P_{k+1}$ is V_i crossing $\}$. Then we associate with P_{k+1} the $(\ell+1)$-tuple $(\ell; j_m, j_{m-1}, \dots, j_{m-\ell+1})$, where for $m - \ell < i \le m$ we define j_i by: $P_{k+1} \subseteq P^{j_i}_{m+v-1,i}$. (For $\ell = 0$ we get merely (0)). We call this the signature of P_{k+1} in P_{m+v} with respect to $(V_1, V_2, \dots, V_m)$. An r-coloring of the $(k+1)$-parameter subsets of P_{m+v} will be called a $(V_1, V_2, \dots, V_m)$-coloring if the colors of all $(k+1)$-parameter subsets with the same signature are the same.

We next present an iterated form of Lemma 1. For arbitrary positive integers m and v, define the integers v_i, $1 \le i \le m$ as follows (where the various values of the function N exist by the induction hypothesis of the theorem):

$$v_1 = N(L, \bar{C}, H, k, r^{t^{m-1}}, v, \dots, v),$$

$$v_2 = N(L, \bar{C}, H, k, r^{t^{m-2}}, v_1 + 1, \dots, v_1 + 1),$$

$$\vdots$$

$$v_{i+1} = N(L, \bar{C}, H, k, r^{t^{m-i-1}}, v_i + 1, \dots, v_i + 1),$$

$$\vdots$$

$$v_m = N(L, \bar{C}, H, k, r^{t^0}, v_{m-1} + 1, \dots, v_{m-1} + 1).$$

Lemma 2: Let m and v be positive integers. Let $P_x = P(A,B,H,\Pi,f,w,x)$ be an x-parameter set in A^w with $x \ge v_m$. Suppose the $(k+1)$-parameter subsets of P_x are r-colored.

Then P_x contains an $(m+v)$-parameter subset P_{m+v}, with partition $\{S_0, S_1, \dots, S_v, V_1, \dots, V_m)$, such that the r-coloring restricted to P_{m+v} is a $(V_m, V_{m-1}, \dots, V_1)$-coloring.

With the help of Lemma 2, the theorem can now be proved. We define $v = \max\limits_{1 \le i \le r} N(A, \bar{B}, H, k+1, r, t_1, \dots, t_i - 1, \dots, t_r)$, $m = N(A,B,H,0,r^k,1,1,\dots,1)$, $K = \binom{t^v}{t^{k+1}}$, and we let $v_1, \dots, v_m$

be as previously defined. The induction step can be shown to hold for the choice $N(A,\bar{B},H,k+1,r,t_1,\ldots,t_r) = v_m$.

A very informal sketch of the remainder of the proof can be given as follows. The original r-coloring of the $(k+1)$-parameter subsets of an N-parameter set P (with $N \geq v_m$) induces an r-coloring of the k-parameter subsets of the induced $(N-1)$-parameter set P^*. By the induction hypothesis we can find a "large" parameter set $Q^* \subseteq P^*$ with all k-parameter subsets of Q^* the same color.

Using the map M, we can show that this induces a large parameter set $Q \subseteq P$ such for some "direction" all the crossing $(k+1)$-parameter subsets of Q are one color. The remaining $(k+1)$-parameter subsets of Q fall into t classes which are translates of one another such that each class consists of the set of $(k+1)$-parameter subsets of a parameter subset Q' of Q. We choose one of these t classes, say, the class of $(k+1)$-parameter subsets of Q', and recolor each of these $(k+1)$-parameter subsets using r^t colors according to the way in which the corresponding translates of the particular $(k+1)$-parameter subset were r-colored. We again go up into the $*$-sets, use the induction hypothesis and the map M, and obtain a large parameter subset R of Q' such that for some new direction all crossing $(k+1)$-parameter subsets of R are one color (probably a different color than that of the first class of $(k+1)$-parameter subsets). As before, all the remaining $(k+1)$-parameter subsets of R fall into t classes which are translates of one another such that each class consists of the set of $(k+1)$-parameter subsets of a parameter subset R' of R. We choose one of these t classes and recolor the $(k+1)$-parameter subsets in this class using r^{t^2} colors according to the way that the corresponding translates are r^t-colored.

We iterate this process for a large number of steps (the exact numbers need not concern us here) and we then are faced with the following interesting configuration. We have a very large parameter set S and a large set of directions so that the color of any $(k+1)$-parameter set which crosses in any of the specified directions depends only on the directions in which it crosses. The $(k+1)$-parameter sets which do not cross in any of these directions naturally fall into parallel classes, each class being the set of $(k+1)$-parameter subsets of some (still large) parameter subset T of S. There is a natural correspondence we can make between this configuration and a large parameter set $\hat{S}$ in which the "points" of the $\hat{S}$ correspond to these various parameter subsets T. These "points" of $\hat{S}$ may be colored according to the way in which the (ordered) set of $(k+1)$-parameter sets in the corresponding subset T are colored (using a number of colors depending only on r, t and the number of parameters in T). By the Theorem for the case $k = 0$, $\ell = 1$ we can extract a 1-parameter subset $\hat{P}_1$ of $\hat{S}$ all of whose 0-parameter subsets are the same color.

This corresponds in S to a set of t "parallel" large parameter subsets T_i, $1 \le i \le t$, all of whose corresponding $(k+1)$-parameter subsets have the same color and such that (by the iterative construction) all $(k+1)$-parameter sets of $\bigcup_i T_i$ which intersect more than one T_i have the same color, say, the i^{th} color. By the choice of v_m, the T_i have so many parameters that either for some $j \neq i$ one of them (and therefore all of them) contains a t_j-parameter subset all of whose $(k+1)$-parameter subsets have the j^{th} color and we are done or each contains a (t_i-1)-parameter subset all of whose $(k+1)$-parameter subsets have the i^{th} color.

However, the union of these t (t_i-1)-parameter sets exactly forms a t_i-parameter set all of whose $(k+1)$-parameter subsets have the i^{th} color and we are also done. This completes the proof of the Theorem.

SOME APPLICATIONS

In this section we present several corollaries to the Theorem, the most well-known of these perhaps being the theorem of van der Waerden on arithmetic progressions (Corollary 7). Other corollaries are new, in particular, the results for affine and vector spaces, which we present first.

The way in which most of the results follow from the Theorem is relatively straightforward and the actual values of the assorted variables needed for the derivations will not be given.

Corollary 1: Let ℓ, r be positive integers, $k = 0$ or 1, and $F = GF(q)$ a finite field. Then there is an integer $N = N(q, r, \ell, k)$, depending only on q, r, ℓ and k with the following property: If A is an affine space over F of dimension $n \geq N$, and if all the k-dimensional affine subspaces of A are r-colored in any way, then there is some ℓ-dimensional affine subspace of A with all of its k-dimensional affine subspaces the same color.

Corollary 2: Let ℓ, r be positive integers, $F = GF(q)$ a finite field and $k = 0$ or 1. Then there is a number $N = N(q, r, \ell, k)$, depending only on q, r, ℓ and k with the following property: If V is an n-dimensional vector space with $n \geq N$, and if the k-dimensional vector subspaces of V are r-colored in any way, then there is an ℓ-dimensional vector subspace of V with all of its k-dimensional vector subspaces one color.

Remark: The last corollary (Rota's conjecture for $k = 0,1$) is also true for $k = 2$. This result is not a direct corollary of the Theorem, but follows from Corollary 1 by an inductive argument, which can be found in [8].

This argument, in fact, shows that if the affine analogue is true for some fixed k, and all q, r, ℓ, then Rota's conjecture is true for $k+1$, and all q, r, ℓ.

Corollary 3: Given integers ℓ and r, there exists an integer $N(\ell, r)$ such that if S is a finite set with $|S| \geq N(\ell, r)$ and the subsets of S are r-colored, then there exist ℓ disjoint nonempty subsets $S_1, \dots, S_\ell$ of S such that all $2^\ell - 1$ unions $\bigcup_{j \in J} S_j$, $\emptyset \neq J \subseteq \{1, 2, \dots, \ell\}$, are one color.

Corollary 4: (J. Folkman, J. Sanders [10], R. Rado [6]) Given integers ℓ and r, there exists an integer $N'(\ell, r)$ such that if $n \geq N(\ell, r)$ and the positive integers $\leq n$ are r-colored then there exists ℓ integers

$a_1, \ldots, a_\ell$ such that all the sums $\{ \sum_{i=1}^{\ell} \varepsilon_i a_i : \varepsilon_i = 0$ or 1, not all $\varepsilon_i = 0 \}$ are one color.

The case $\ell = 2$ or Corollary 4 was first proved by Schur [11]. Corollary 4 is actually a special case of the following corollary, which is contained in a result of R. Rado [5].

Corollary 5: Let $\mathcal{L} = L_i(x_1, \ldots, x_m)$, $1 \le i \le n$, be a system of homogeneous linear equations with real coefficients with the propertv that for each j, $1 \le j \le m$, there exists a solution $(\varepsilon_1, \ldots, \varepsilon_m)$ to the system $\mathcal{L}$ with $\varepsilon_i = 0$ or 1 and $\varepsilon_j = 1$. Then given an integer r there exists an integer $N(r)$ such that if $n \ge N(r)$ and the positive integers $\le n$ are r-colored, then $\mathcal{L}$ can be solved with integers of one color.

By a multigrade of order m we mean two disjoint sets of integers $\{c_i\}$, $\{d_i\}$, $1 \le i \le n$, such that

$$\sum_{i=1}^{n} c_i^k = \sum_{i=1}^{n} d_i^k, \quad \text{for} \qquad k = 1, 2, \ldots, m.$$

We denote this by

$$c_1, \ldots, c_n \overset{m}{=} d_1, \ldots, d_n$$

Since $\{ac_i + b\}$, $\{ad_i + b\}$, $1 \le i \le m$, is a multigrade of order n if $\{c_i\}$, $\{d_i\}$, $1 \le i \le n$, is, then a straightforward application of the Theorem along the lines used in the preceding corollaries yields

Corollary 6: If the multigrade equations

$$(*) \qquad x_1, \ldots, x_n \overset{m}{=} y_1, \ldots, y_n$$

have any integer solution (which always happens, for example, if $n \ge 2^{m-1}$), then for any r-coloring of the positive integers, $(*)$ always has a solution in integers of one color.

Corollary 7: (van der Waerden [4, 12]) Given integers t and r, there exists an integer $M(t,r)$ such that if $n \geq M(t,r)$ and the non-negative integers $< n$ are arbitrarily r-colored, then there must exist a monochromatic arithmetic progression of length t.

This result is implied by the stronger

Corollary 8: (Hales-Jewett [3]) Let $A = \{a_1, \ldots, a_t\}$ be a finite set. Given an integer r there exists an integer $N(r,t)$ such that if $n \geq N(r,t)$ and the set A^n is r-colored then there exists a set of t elements of A^n of the form

$$X_i = (x_{11}, \ldots, x_{1u}, a_i, x_{21}, \ldots, x_{2v}, a_i, \ldots, a_i, x_{d1}, \ldots, x_{dz}) \in A^n, \quad 1 \leq i \leq t,$$

all of which have the same color.

We conclude with a final (stronger) application of the Theorem.

Let $C_n = \{(x_1, \ldots, x_n) : x_i = 0 \text{ or } 1\}$ be the set of 2^n vertices of a unit n-cube in $\mathbb{R}^n$. Let us call a subset $Q_k \subseteq C_n$ a k-subspace of C_n if $|Q_k| = 2^k$ and Q_k is contained in some k-dimensional euclidean subspace of $\mathbb{R}^n$.

Corollary 9: Given integers k, ℓ, r, there exists an integer $N(k,\ell,r)$ such that if $n \geq N(k,\ell,r)$ and the k-subspaces of C_n are r-colored, then there exists an ℓ-subspace of C_n all of whose k-subspaces are one color.

CONCLUDING REMARKS

Several questions come to mind at this point.

(i) In the corollaries of the Theorem listed, we never really make much use of the freedom we have in choosing B and H. What are some interesting applications for some less trivial choices of B and H?

(ii) Are the various infinite versions of certain of the corollaries valid? A specific simple case would be: If the positive integers are 2-colored, is it true that there always exists an infinite subset A such that all sums

$$\sum_{b \in B} b, \qquad \phi \neq B \subseteq A, \qquad B\text{-finite},$$

are one color?

(iii) The reader will notice that the original theorem of Ramsey (for subsets of finite set) does not appear as a corollary to the Theorem. Is there a more general theorem which includes both of these results in a natural way? In a certain sense, Ramsey's theorem for sets corresponds to taking $t = 1$ in the Theorem (something which we are prohibited from doing), much in the spirit found in the paper of Goldman and Rota [1] on finite vector spaces.

(iv) With respect to the corollaries, the upper bounds given by the Theorem on the various N's are rather crude, to say the least. Is it possible to improve the estimates of these numbers? For example, in Corollary 9, the upper bound on $N(1,2,2)$ given by the Theorem is truly enormous, where, in fact, the exact bound is probably < 10.

(v) It was suggested by M. Simonovits that perhaps it would be possible to give an intrinsic definition of k-parameter sets, i.e., one which does not depend on coordinates. If this is possible then conceivably the corresponding proofs might become simpler.

(vi) Our particular definition of k-parameter set was chosen, to a certain extent, because a Ramsey theorem for them could be proved. What other definitions will have this property? In particular, can a suitable one be found which will establish Rota's original conjecture for k-subspaces of finite vector space, $k \geq 3$?

REFERENCES

[1] GOLDMAN, J. and ROTA, G. C., Combinatorial theory of finite vector spaces, (to appear).

[2] GRAHAM, R. L. and ROTHSCHILD, B. L., Ramsey's Theorem for n-dimensional arrays, Bull. A.M.S. 75 (1969) 418-422.

[3] HALES, A. W. and JEWETT, R. I., Regularity and positional games, Trans. A.M.S. 106 (1963) 222-229.

[4] KHINCHIN, A. Y., Three pearls of number theory, Graylock Press, Rochester, 1952.

[5] RADO, R., Note on combinatorial analysis, Proc. London Math. Soc. (2) 48 (1943) 122-160.

[6] RADO, R., see paper in this Proceedings.

[7] RAMSEY, F. P., On a problem of formal logic, Proc. London Math. Soc., 2nd ser., 30 (1930) 264-286.

[8] ROTHSCHILD, B. L., A generalization of Ramsey's theorem and a conjecture of Rota, Doctoral Dissertation, Yale University, New Haven, Connecticut, 1967.

[9] RYSER, H. J., Combinatorial Mathematics, Wiley, New York, 1963.

[10] SANDERS, J., A generalization of a theorem of Schur, Doctoral Dissertation, Yale University, New Haven, Connecticut, 1968.

[11] SCHUR, I., Über die Kongruenz $x^m + y^m \equiv z^m \pmod p$, Jahr. Deutsch. Math. - Verein. 25 (1916), 114.

[12] VAN DER WAERDEN, B. L., Beweis einer Baudetschen Vermutung, Nieuw Arch. Wiskunde 15 (1927) 212-216.