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A CONSTRUCTIVE SOLUTION TO A TOURNAMENT PROBLEM

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Introduction. By a *tournament* T_n *on n vertices*, we shall mean a directed graph on n vertices for which every pair of distinct vertices form the endpoints of exactly one directed edge (e.g., see [5]). If x and y are vertices of T_n we say that x *dominates* y if the edge between x and y is directed from x to y. In 1962, K. Schütte [2] raised the following question: Given $k>0$, is there a tournament $T_{n(k)}$ such that for any set S of k vertices of $T_{n(k)}$ there is a vertex y which dominates all k elements of S. (Such a tournament will be said to have *property* P_k.)

In [3], P. Erdös showed by probabilistic arguments that for each k, such a $T_{n(k)}$ must exist. Thus, it is meaningful to define $f(k)$ to be the minimum value of $n(k)$ for which such a $T_{n(k)}$ exists. More precisely, Erdös showed that

$$f(k) \le k^2 2^k (\log 2+\varepsilon) \tag{1}$$

for any $\varepsilon>0$ provided k is sufficiently large. In the other direction Szekeres and Szekeres [6] established

$$f(k) \ge (k+2)2^{k-1}-1. \tag{2}$$

In this note, we give for each k an *explicit construction* of a tournament $T_{n(k)}$ which has property P_k. Although the best bound we currently have on the value of $n(k)$ needed by our construction shows that $n(k)$ may be as large as $k^2 2^{2k-2}$, in fact, for small values of k, our tournaments are minimal.

Construction of the tournament. Let p be a prime congruent to 3 modulo 4 and let $\{0, 1, \ldots, p-1\}=V$ be the set of vertices of T_p. Define the edges of T_p by directing an edge from i to j iff $i-j$ is a quadratic residue of p, i.e., iff $\left(\frac{i-j}{p}\right)=1$, where we use the familiar Legendre symbol (cf. [4]). Since $p\equiv 3 \bmod 4$ then $\left(\frac{-1}{p}\right)=-1$ so that any two distinct vertices are joined by exactly one edge and T_p is a well-defined tournament.

THEOREM. *If* $p>k^2 2^{2k-2}$ *then* T_p *has property* P_k.

Proof. It is easily seen that T_p has property P_k iff for all $a_1, \ldots, a_k \in V$,

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there exists an $x \in V$ such that $\left(\frac{x-a_i}{p}\right)=1$ for $1 \le i \le k$. Set $\chi(a)=\left(\frac{a}{p}\right)$ and let $A=\{a_1, \ldots, a_k\}$ denote a set of k arbitrary fixed elements of V. Define $g(A)$ by

(3) $$g(A) = \sum_{\substack{x=0 \\ x \notin A}}^{p-1} \prod_{j=1}^{k} [1+\chi(x-a_j)].$$

If we can show $g(A)$ is always >0 then the theorem is proved; for, in this case, there is a choice $x=x_0 \notin A$ such that $\prod_{j=1}^{k} [1+\chi(x_0-a_j)]>0$ and, hence, $\chi(x_0-a_j) \neq -1$ for $1 \le j \le k$. Since $x_0 \notin A$, then $x_0-a_j \neq 0$ and $\chi(x_0-a_j)\neq 0$. Thus, $\chi(x_0-a_j)=1$ for $1 \le j \le k$ and by the previous remark, we would be done.

We next show $g(A)>0$. Define $h(A)$ by

(4) $$h(A) = \sum_{x=0}^{p-1} \prod_{j=1}^{k} [1+\chi(x-a_j)].$$

Thus,

(5) $$g(A) = h(A)- \sum_{i=0}^{k} \prod_{j=1}^{k} [1+\chi(a_i-a_j)].$$

Expanding the inner terms in (4) we obtain

(6) $$\begin{aligned} h(A) = \sum_{x=0}^{p-1} 1 + \sum_{x=0}^{p-1} \sum_{j=1}^{k} \chi(x-a_j) + \sum_{x=0}^{p-1} \sum_{j_1<j_2} \chi(x-a_{j_1})\chi(x-a_{j_2}) + \cdots \\ \cdots + \sum_{x=0}^{p-1} \sum_{j_1<\cdots<j_s} \chi(x-a_{j_1})\ldots\chi(x-a_{j_s}) + \cdots \\ \cdots + \sum_{x=0}^{p-1} \sum_{j_1<\cdots<j_k} \chi(x-a_{j_1})\ldots\chi(x-a_{j_k}). \end{aligned}$$

The first two terms of (6) are p and 0 respectively. To estimate the remaining terms we rely on the following powerful result of D. A. Burgess [1]:

(7) $$\left|\sum_{x=0}^{p-1} \chi(x-a_{j_1})\ldots\chi(x-a_{j_s})\right| \le (s-1)\sqrt{p}$$

for $a_{j_1}, \ldots, a_{j_s}$ distinct. Thus, we have

(8) $$\left|\sum_{x=0}^{p-1} \sum_{j_1<\cdots<j_s} \chi(p-a_{j_1})\ldots\chi(x-a_{j_s})\right| \le \binom{k}{s}(s-1)\sqrt{p}$$

and therefore

(9) $$|h(A)-p| \le \sqrt{p} \sum_{s=2}^{k} \binom{k}{s}(s-1).$$

A straightforward calculation shows

(10) $$\sum_{s=2}^{k} \binom{k}{s}(s-1) = (k-2)2^{k-1}+1$$

so that we have

$$h(A) \geq p-[(k-2)2^{k-1}+1]\sqrt{p}. \tag{11}$$

Now consider the expression

$$\sum_{i=0}^{k} \prod_{j=1}^{k} [1+\chi(a_i-a_j)] = h(A)-g(A)$$

which occurs in (5). If $h(A)-g(A)\neq 0$ then for some i_0 the product $\prod_{j=1}^{k} [1+\chi(a_{i_0}-a_j)]$ is nonzero. Thus, for all j, $\chi(a_{i_0}-a_j)\neq -1$ so that for all $j\neq i_0$, $\chi(a_{i_0}-a_j)=1$. But this implies $\chi(a_j-a_{i_0})=-1$ for all $j\neq i_0$ and consequently

$$\prod_{j=1}^{k} [1+\chi(a_i-a_j)] = \begin{cases} 0 & \text{for } i \neq i_0 \\ 2^{k-1} & \text{for } i = i_0. \end{cases} \tag{12}$$

Therefore, in any case, we have

$$h(A)-g(A) \leq 2^{k-1}. \tag{13}$$

Applying (11) we obtain

$$g(A) \geq p-[(k-2)2^{k-1}+1]\sqrt{p}-2^{k-1}. \tag{14}$$

It is easily checked that for $p>k^2 2^{2k-2}$, the right-hand side of (14) is >0. This proves the theorem.

Concluding remarks. The value $k^2 2^{2k-2}$ is nearly the square of the nonconstructive upper bound (1) of Erdös. Specific constructions show that much smaller values p suffice to endow T_p with property P_k. For example, T_7 has property P_2 and T_{19} has property P_3. In [6] it is shown that $f(2)=7$ and $f(3)=19$ so that these tournaments are minimal. Also, it is true that T_{67} has property P_4. Since (2) gives $f(4)\geq 47$ it is possible that T_{67} is also minimal.

If q is an odd power of a prime congruent to 3 modulo 4 then T_q can be defined with vertices as elements of $GF(q)$ and an edge directed from i to j iff $i-j$ is a square in $GF(q)$. It can be shown for example that T_{27} has property P_3. However, no examples are known for which the number of vertices of a T_q with property P_k is smaller than a suitable T_p.

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