ROTA'S GEOMETRIC ANALOGUE TO RAMSEY'S THEOREM

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1. Let $L=\{L_i \mid i=0, 1, 2, \ldots\}$ be a class of geometric lattices. For integers $k\geq 0$, $r\geq 0$, $t>0$ consider the statement:

> $L(k, r, t)$. There is an integer $N=N_1(k, r, t)$, depending only on k, r, t, such that if $n\geq N$, and if the elements of L_n of rank r are colored with t colors, then there is an element x of rank k such that all the elements y of rank r with $y\leq x$ have the same color.

If we let $L_i=L(S_i)$, the subset lattice of a set S_i of i elements, then this statement, which we denote in this case by $S(k, r, t)$, becomes Ramsey's theorem for k, r, t.

Rota has conjectured that if one chooses the L_i to be $P_i(q)$, the lattice of subspaces of an i-dimensional vector space over GF(q) (or equivalently, the lattice of projective subspaces of an $(i-1)$-dimensional projective space), then the corresponding statement, denoted in this case by $P_q(k, r, t)$, is also true. The conjecture is true for $r=1$ and any k, t and q. We will indicate part of the proof here. Details will appear elsewhere.

2. First we consider another statement, namely $A_q(k, r, t)$, by which we mean $L(k, r, t)$ with $L_i=A_i(q)$, the subspace lattice of an affine $(i-1)$-dimensional space over GF(q). Using the well-known relationship between the affine and projective lattices (see the lemma below) we reduce P_q to A_q, of which the case $r=1$ is proved.

There are in fact three results we can obtain from the relationship, namely:

THEOREM 1. $P_q(k, r, t) \Rightarrow A_q(k, r, t)$.

THEOREM 2. $A_q(k+1, r+1, t) \Rightarrow P_q(k, r, t)$.

THEOREM 3. $\forall k\, A_q(k, r, t) \Rightarrow \forall k\, P_q(k, r, t)$.

Theorems 1 and 2 provide information about the relationship of the corresponding numbers N for the affine and projective cases. But it is clearly Theorem 3 that is necessary to reduce the projective to the affine problem for $r=1$, and thus we sketch a proof of Theorem 3 below.

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3. LEMMA. *Let x be a dual atom of $P_n(q)$ (i.e. an $(n-2)$-dimensional hyperplane). Let $P_{n-1}=\{y \mid y\leq x\}$, $A_n=\{y \mid y\nleq x$ or $y=0\}$. Then*:

(a) *P_{n-1} with the induced order is isomorphic to $P_{n-1}(q)$.*

(b) *A_n with the induced order is isomorphic to $A_n(q)$.*

(c) *For each $y\in P_{n-1}$, there is a $z\in A_n$ with $y\leq z$ and the rank of z one greater than the rank of y.*

(d) *For each $z\neq 0$, $z\in A_n$, the rank of $z\wedge x$ is one less than the rank of z.*

4. We now indicate a proof of Theorem 3. Assume $A_q(k, r, t)$ for all k. Let l_1 be a large integer. Then the lemma and $A_q(l_1, r, t)$ imply that if we color with t colors all the rank r elements of $P_n(q)$ for sufficiently large n, then (with A_n, x and P_{n-1} as in the lemma) there is an element of u_1 of rank l_1 in A_n such that all rank r elements y of A_n with $y\leq u_1$ have the same color. That is, $P_n(q)$ contains an element u_1 of rank l_1 such that when one divides $P_{l_1}=\{y \mid y\leq u_1\}$ into $P_{l_1-1}=\{y \mid y\leq u_1\wedge x\}$ and $A_{l_1}=\{y \mid y\leq u_1, y\nleq u_1\wedge x$ or $y=0\}$ as in the lemma, then all the rank r elements of A_{l_1} have the same color. By the lemma P_{l_1-1} is isomorphic to $P_{l_1-1}(q)$ and A_{l_1} to $A_{l_1}(q)$. Hence we can apply these same arguments to P_{l_1-1} instead of $P_n(q)$.

So if we let l_2 be a large integer, and if l_1 is sufficiently large, then P_{l_1-1} contains an element u_2 of rank l_2 such that $P_{l_2}=\{y \mid y\leq u_2\}$ is isomorphic to $P_{l_2}(q)$, and it is divided into P_{l_2-1} and A_{l_2}, as in the lemma, with all rank r elements of A_{l_2} having the same color (but not necessarily the same as the color for A_{l_1}). (See Figure 1.)

We repeat this argument, say, $m=k_0(t-1)+1$ times, for an arbitrary k_0. Then this gives a sequence of pairs

$$(A_{l_1}, P_{l_1-1}), (A_{l_2}, P_{l_2-1}), \ldots, (A_{l_m}, P_{l_m-1}),$$

where $P_{l_i-1}\supseteq A_{l_{i+1}}\cup P_{l_{i+1}-1}$, and all the rank r elements of A_{l_i} have the same color (depending on i). But since there are only t colors, then one of them must occur k_0 times. So by renumbering, we obtain a sequence

$$(A_{m_1}, P_{m_1-1}), (A_{m_2}, P_{m_2-1}), \ldots, (A_{m_{k_0}}, P_{m_{k_0}-1})$$

with $P_{m_i-1}\supseteq A_{m_{i+1}}\cup P_{m_{i+1}-1}$, and with all the elements of any of the A_{m_i} of rank r having the same color.

Now we use part (c) of the lemma to find elements $a_1\in A_{m_{k_0}}$, $a_2\in A_{m_{k_0}-1}, \ldots,$ $a_{k_0}\in A_{m_1}$ with $a_i>a_{i-1}$ for all i, and each a_i of rank i. Using part (d) of the lemma, we see that any y of rank r with $y\leq a_{k_0}$ is in A_{m_i} for some i. Hence all such y have the same color, and the element a_{k_0} establishes $P_q(k_0, r, t)$. Since k_0 was arbitrary, Theorem 3 is proved.

5. We note that if we consider $L(S_n)$ instead of $P_n(q)$ in the lemma, $L(S_{n-1})$ instead of P_{n-1}, and $L'(S_{n-1})$ instead of $A_n(q)$, where $L'(S_{n-1})$ is $L(S_{n-1})$ with an extra element appended below everything else, then the statements (a), (b), (c), (d) are still true. So the proof of Theorem 3 is still valid. But since coloring rank r

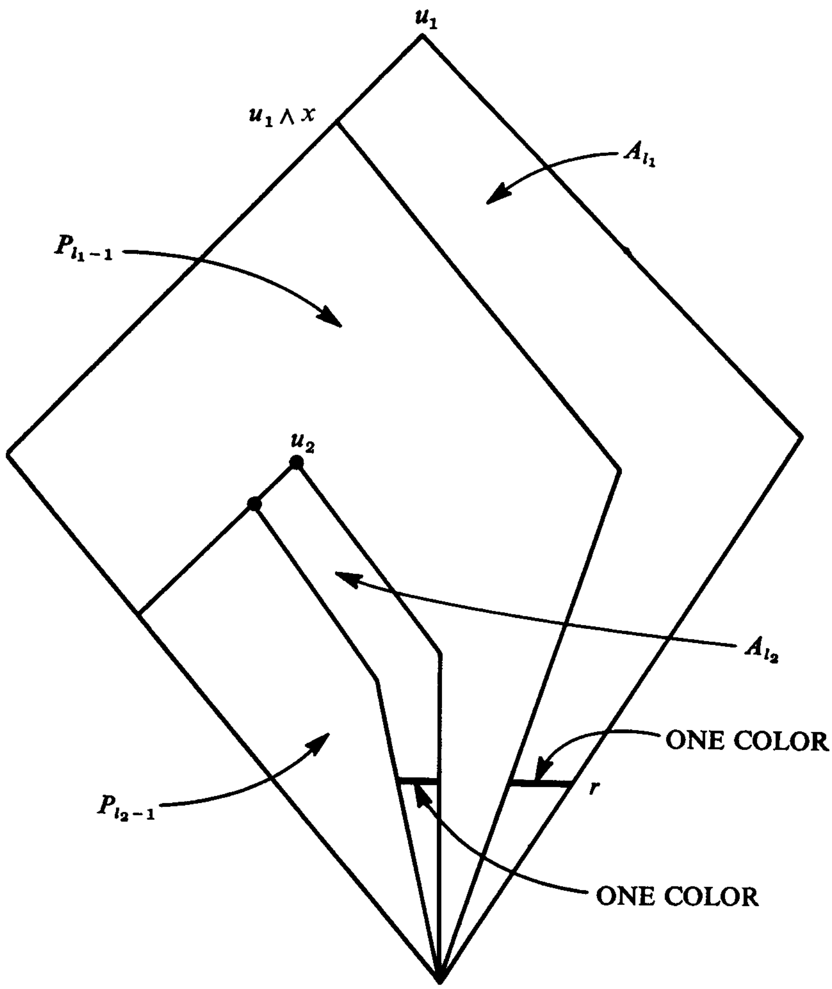

FIGURE 1

elements of $L'(S_{n-1})$ is equivalent to coloring rank $r-1$ elements of $L(S_{n-1})$, we obtain:

THEOREM 3'. $\forall k\, S(k, r-1, t) \Rightarrow \forall k\, S(k, r, t)$.

This is just the induction step in the proof of Ramsey's theorem.

6. Finally, we state the result from which one proves $A(k, 1, t)$ for all k.

THEOREM 4. *Let F be a finite set, and let $A=\{A_1, \ldots, A_r\}$ be a set of m-lists of elements of F. For each t there is a number $N=N(t, r, m)$ such that for $n \geq N$ and any* *coloring of the n-lists of F with t colors there are numbers $m_1, m_2, \ldots, m_d$, $d \geq 2$, and n-lists*

$$B_i = (x_{11}, \ldots, x_{1m_1}, A_i, x_{21}, \ldots, x_{2m_2}, A_i, \ldots, x_{d-1,m_{d-1}}, A_i, x_{d1}, \ldots, x_{dm_d}), \quad i = 1, 2, \ldots, r,$$

which all have the same color.

This result somewhat generalizes one of Hales and Jewett [1]. It uses arguments exactly like those used in proving van der Waerden's theorem. In fact both $A(k, 1, t)$ and van der Waerden's theorem are immediate corollaries of Theorem 4. To get $A(k, 1, t)$, we let $F = \mathrm{GF}(q)$ and $A = \{\text{all } (k-1)\text{-lists of } F\}$. Then the B_i of Theorem 4 are the points $(r=1)$ of an affine subspace of dimension $k-1$. To get van der Waerden's theorem, let $F = \{0, 1, 2, \ldots, l-1\}$, and let $m=1$, $A = \{0, 1, \ldots l-1\}$. Then if we think of n-lists as representations of integers in base l, the B_i form a length l arithmetic subprogression.

References

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