A SURVEY OF FINITE RAMSEY THEOREMS

by

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INTRODUCTION

Almost everyone is familiar with the fact that any way of coloring the edges of the complete graph on 6 vertices with 2 colors always results in a triangle having all its edges the same color. This is a very special case of a more general result known as Ramsey's Theorem, first proved in 1930 [21],[23], which can be stated as follows: For any positive integers k, ℓ and r and any r-coloring* of the k-subsets† of a sufficiently large n-set S, all the k-subsets of some ℓ-subset of S have one color.

In another direction, I. Schur proved in 1916 [25] that for any r and any r-coloring of a sufficiently large initial segment of the positive integers, one can always solve the equation x + y = z with integers having one color.

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*i.e., partitioning into r classes.

†i.e., subsets with k elements.

These two theorems are typical of what we shall call a Ramsey theorem and a Schur theorem, respectively. In this paper we shall survey a number of more general Ramsey and Schur theorems which have appeared in the past 40 years. It will be seen that quite a few of these results are rather closely related, e.g., van der Waerden's theorem on arithmetic progressions [26],[15], Rado's work on regularity and systems of linear equations [19],[18], the results of Hales and Jewett [13] and others [7] on arrays of points and Rota's conjectured analogue of Ramsey's Theorem for finite vector spaces, as well as the original theorems of Ramsey and Schur.

NOTATION

Given a set S, by an r-coloring of S we mean a partition of S into r (possibly empty) subsets S_i, $1 \le i \le r$. A subset X of S is said to be monochromatic if $X \subseteq S_i$ for some i.

A rank function ρ on a partially ordered set P is a function from P to $\{0,1,2,\dots\}$. If $a \in P$ and $\rho(a) = k$, then a is said to have rank k. P is graded by ρ if for all $a,b \in P$ such that $a \le x \le b$ implies $a = x$ or $b = x$, we have $\rho(b) = \rho(a) + 1$. By a Ramsey theorem $\mathcal{R}(P)$ for P we mean the following:

$\mathcal{R}(P)$: Given positive integers k, ℓ and r, there exists an integer $N = N(k,\ell,r)$ such that if $n \ge N$ and all the rank k elements which are below a rank n element p of P are r-colored then all the rank k elements which are below <u>some</u> rank ℓ element below p have one color.

Similarly, if we have a <u>product</u> defined on P, i.e., a mapping of P×P into P which we shall assume is <u>associative</u> then we can state a Schur theorem $\mathcal{S}(P)$ for P as follows:

$\mathcal{S}(P)$: Given positive integers k and r, there exists an integer $N = N(k,r)$ such that if $n \geq N$ and all the elements which are below a rank n element p of P are r-colored then there exist k distinct element $p_1, \ldots, p_k$ below p such that all nonempty products formed from distinct p_i are below p and have one color.

We next discuss some partially ordered sets P for which $\mathcal{R}(P)$ or $\mathcal{S}(P)$ has been established and we indicate their mutual interrelations. For the following section, the reader is referred to Diagram 1.

SOME RAMSEY AND SCHUR THEOREMS

(<u>GLR</u>). This is a very general Ramsey theorem for graded partially ordered sets (or equivalently, certain classes of categories) recently proved by Graham, Leeb and Rothschild [8]. Essentially, it is shown that if a class of graded partially ordered sets satisfies four rather strong axioms, then a Ramsey theorem is valid for the class. The precise statements involved are somewhat technical and will

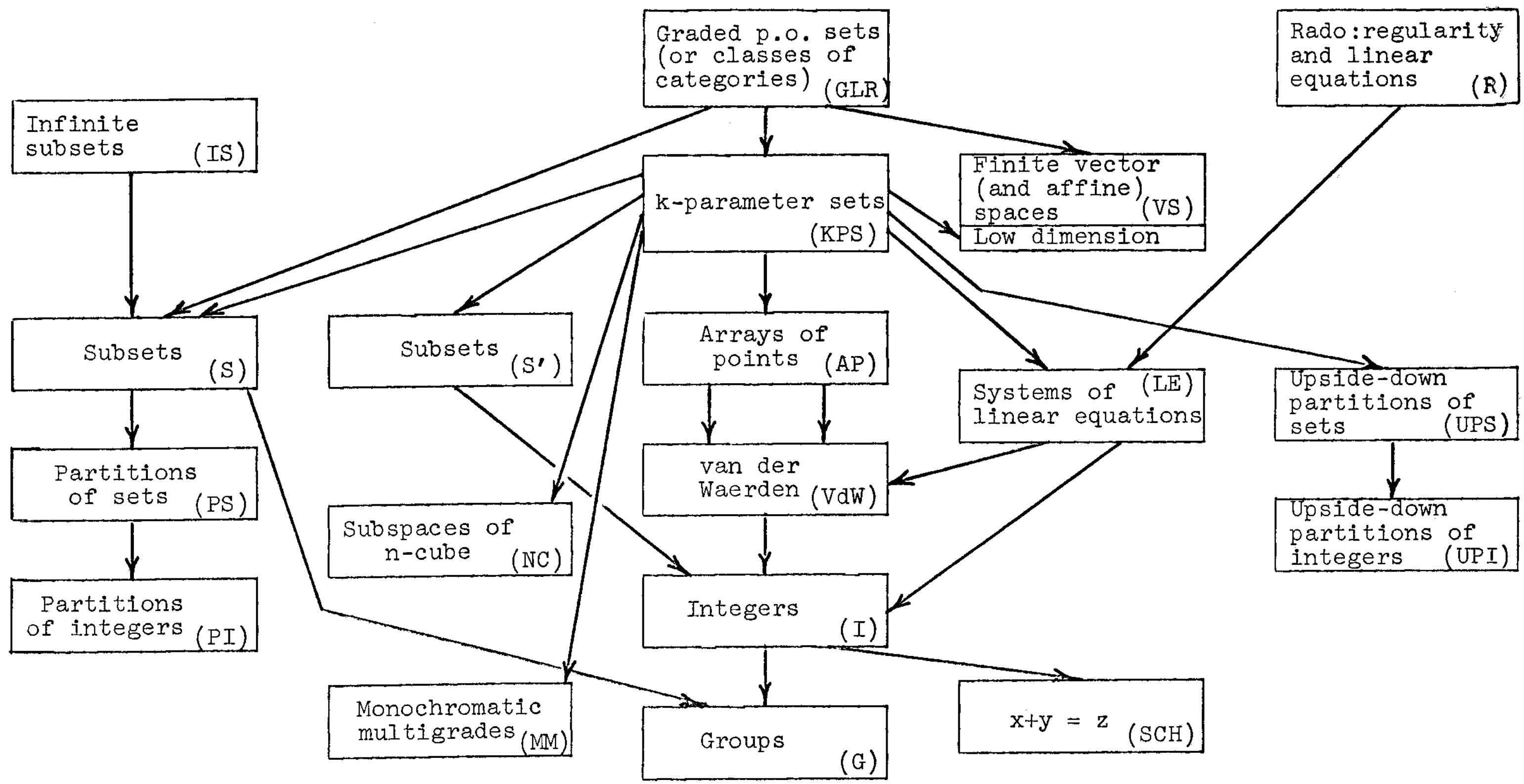

Diagram 1

Interrelations of the Theorems

not be discussed here. As can be seen from the diagram, (GLR) implies nearly all of the other results to be described in the paper.

(<u>VS</u>). Suppose V is a countably infinite dimensional vector space over a fixed finite field F. Let P denote the graded partially ordered set of all finite dimensional subspaces of V, partially ordered by inclusion with rank equal to dimension. The Ramsey theorem $\mathfrak{R}(P)$, first conjectured by Rota, can be deduced directly from (GLR).

If the notion <u>vector</u> space is replaced by that of <u>affine</u> space, the corresponding graded partially ordered set P′ satisfies the Ramsey theorem $\mathfrak{R}(P')$. This is also a consequence of (GLR). Prior to (GLR), it had been shown by Graham and Rothschild [9] that $\mathfrak{R}(P)$ and $\mathfrak{R}(P')$ were equivalent. The special case k = 1 for $\mathfrak{R}(P)$ was previsouly proved by Kleitman [16] for F having 2 elements, by Rothschild [22] for F having 2, 3 and 4 elements and by Graham and Rothschild [11],[10] for all F. (This is indicated by the arrow in the diagram from KPS to the "low dimension" part of VS.)

(<u>KPS</u>). The concept of k-parameter set was introduced by Graham and Rothschild in [11]. Basically, k-parameter sets P_k are certain distinguished subsets of A^n where A is a fixed finite set and n is an integer.

P_k also depends upon a given permutation group acting on A and several other variables; the reader is referred to [11] for a detailed discussion. In particular, it is possible to form a graded partially ordered set P from certain classes of k-parameter sets (with variable k) with the rank of P_k equal to k and partially ordered by inclusion. The Ramsey theorem $\mathfrak{R}(P)$ for P, established in [11], has proved quite useful in deriving and interrelating a number of results in this field.

(S). This, of course, is just the original theorem of Ramsey [21] with P being the set of all finite sets of some countable set partially ordered by inclusion and with rank equal to cardinality. The two arrows (GLR) $\rightarrow$ (S) and (KPS) $\rightarrow$ (S) in the diagram indicate that the corresponding derivations given in [8] and [11] respectively are essentially different.

(S) can also be deduced from the following theorem, also due to Ramsey [21]:

(IS). Given k and r, if all the k-subsets of a countably infinite set S are r-colored then there is an infinite subset $S' \subseteq S$ such that all k-subsets of S have one color.

We shall return to the topic of infinite versions of Ramsey and Schur theorems at the end of the paper.

(PS). For a fixed countably infinite set S, we consider the set P of all partitions of finite subsets of S, partially ordered by refinement. The rank of a partition $T = T_1 + \dots + T_t$ is defined to be $|T| - t$. We note that the Ramsey theorem $\mathfrak{R}(P)$ is equivalent to (S) by noting that P contains arbitrarily large lower ideals isomorphic to the lower ideals of the subset lattices which occur in (S). (Specifically, one can consider the set of refinements of a partition $T = T_1 + \dots + T_t$ with $|T_k| \leq 2$ for all k.)

By taking S to be the nonnegative integers and associating with each finite subset T of S the integer obtained by interpreting the characteristic function of T as the binary expansion of an integer in the natural way, the Ramsey theorem (PI) follows at once. This is just $\mathfrak{R}(P')$ where P'is the set of partitions of positive integers partially ordered by refinement and with rank defined as in (PS).

(UPS). In this Ramsey theorem, we essentially turn the partially ordered set in (PS) upside down. More precisely we consider the set of partitions P of a countably infinite set S. If $T = T_1 + \dots + T_t$ is a partition then the rank of T is defined to be t. If T and T' are partitions then P is partially ordered by $\leq$, where $T \leq T'$ if and only if T' is a refinement of T. The first proof of the Ramsey theorem $\mathfrak{R}(P)$ was given by Graham and Rothschild in [11]. No simple proof of (UPS) is known.

(UPI). The Ramsey theorem (UPI), for the so-called upside-down partially ordered set of partitions of integers stands in relation to (UPS) in exactly the same way that (PI) is related to (PS). For further details, the reader is referred to [11].

(AP). Let A be a fixed finite set. Given an integer r there exists an integer N(r) such that for any $n \geq N(r)$ and any r-coloring of the n-tuples A^n, there exists a monochromatic set T consisting of $|A|$ elements of A^n of the form

$$T = \{(a_1, \ldots, a_{i_1-1}, x, a_{i_1+1}, \ldots, a_{i_d-1}, x, a_{i_d+1}, \ldots) : x \in A\}.$$

The first published proof of (AP) was given by Hales and Jewett [13]. (AP) follows from (KPS) by choosing $k = 0$, $\ell = 1$ and taking the permutation group on A to be trivial (cf. [11]).

(VdW). A classic theorem of van der Waerden [26], [15] is the following:

(VdW) Given k and r there exists an integer N(k, r) such that any r-coloring of the first N(k, r) positive integers must result in a monochromatic arithmetic progression of k terms.

It is not difficult to see that by choosing A to be $\{0, 1, \ldots, k-1\}$ in (AP) and interpreting n-tuples from A^n as integers to the base k, the monochromatic set T guaranteed by (AP) corresponds to a monochromatic arithmetic progression of k terms. Thus, (AP) implies (VdW). A different proof of this implication may be found in [13].

(<u>S′</u>). We come to the first of the Schur theorems. Let P be the set of all finite subsets of a countably infinite set S partially ordered by inclusion with rank equal to cardinality. Define the "product" of T and T′ to be their union. Then the Schur theorem $\mathcal{S}(P)$ for P holds.

The derivation of (S′) from (KPS) involves using the set $A = \{0,1\}$ in (KPS) and interpreting elements of A^n as characteristic functions of subsets (cf. [11]). In turn, by interpreting the characteristic functions as integers expressed to the base 2, one obtains the Schur theorem $(I) = \mathcal{S}(P')$ for the set P′ of all nonnegative integers partially ordered by size with the rank of k equal to k.

To furnish the reader with an example of the type of argument used in proving the various implications we give a proof due to Folkman [5] that (VdW) implies (I).

We first need some notation. If a and b are integers, [a,b] will denote the integers k with $a \leq k \leq b$. An r-coloring of a set S will be determined by a function $c : S \to [1,r]$. If S is a subset of the integers, P(S) will denote the set of all integers of the form $\sum_{x \in T} x$ where T is a nonempty subset of S. If k and r are positive integers,

$W(k,r)$ denotes the least positive integer such that if c is any coloring of the set $[1,W(k,r)]$ then there are positive integers i, a and d such that $1 \leq i \leq r$, $a + kd \leq W(k,r)$ and $c(a+jd) = i$ for all integers j with $0 \leq j \leq k$. Note that $W(k,r)$ exists by (VdW).

We restate (I) in a form more convenient for this proof.

Theorem (I) (Folkman) Let r be a positive integer and let $t_1, t_2, \ldots, t_r$ be nonnegative integers. There is a positive integer $N = N(t_1, \ldots, t_r)$ such that if c is any r-coloring of $[1,N]$ then there is an integer i and a set of integers S satisfying the following conditions:

(1) $1 \leq i \leq r$;

(2) $|S| = t_i$;

(3) $P(S) \subseteq [1,N]$;

(4) $c(x) = i$ for each $x \in P(S)$.

Proof.

Remark 1. If $r = 1$ we may take $N = \frac{t_1(t_1+1)}{2}$ and conditions (1) - (4) will be satisfied for $i = 1$ and $S = \{1,2,\ldots,t_1\}$.

Remark 2. If $t_j = 0$ for some j with $1 \leq j \leq r$, we may take $N = 1$ and conditions (1) - (4) will be satisfied for $i = j$ and $S = \emptyset$.

Let T_k denote the assertion that the theorem is true whenever $t_1 + t_2 + \ldots + t_r \leq k$. We will prove T_k for all positive integers k by induction on k.

If $t_1 + t_2 + \ldots + t_r \leq 1$ then either $r = 1$ or $t_j = 0$ for some j with $1 \leq j \leq r$. Hence, T_1 follows from Remarks 1 and 2. Now let $k \geq 1$ be an integer and suppose that we have established T_k. Let r be a positive integer and let $t_1, t_2, \ldots, t_r$ be nonnegative integers with $t_1 + t_2 + \ldots + t_r \leq k + 1$. If $t_j = 0$ for some j with $1 \leq j \leq r$ we are done by Remark 2. Hence, we may assume that $t_j > 0$ for $1 \leq j \leq r$. By T_k, for $1 \leq j \leq r$, the required integer $N(t_1, t_2, \ldots, t_j - 1, t_{j+1}, \ldots, t_r)$ exists. Let M be the maximum of these r integers. Let $W = W(M, r)$ and let $N = 2W$.

Suppose that c is an r-coloring of $[1, N]$. Define an r-coloring c' of $[1, W]$ by $c'(x) = c(x+W)$ for each $x \in [1, W]$. By the choice of W there is an integer i with $1 \leq i \leq r$ and positive integers a and d such that $a + Md \leq W$ and $c'(a+kd) = i$ for $0 \leq k \leq M$. For $1 \leq j \leq r$, let $t_j' = t_j$ if $j \neq i$ and $t_i' = t_i - 1$. By the choice of M, $N(t_1', t_2', \ldots, t_r') \leq M$. Hence, if $x \in [1, N(t_1', \ldots, t_r')]$ then $dx \leq dM < a + Md \leq W \leq N$. Therefore, we may define an r-coloring c'' of $[1, N(t_1', \ldots, t_r')]$ by $c''(x) = c(dx)$ for each $x \in [1, N(t_1', \ldots, t_r')]$. By T_k there is an integer j with $1 \leq j \leq r$ and a set of integers T with $|T| = t_j'$ such that $P(T) \subseteq [1, N(t_1', \ldots, t_r')]$ and $c''(x) = j$ for each $x \in P(T)$.

Case 1: $j \neq i$.

Let $S = \{dx | x \in T\}$. Then S is a set of positive integers. $P(S) = \{dx | x \in P(T)\} \subseteq [d, dN(t_1', \ldots, t_r')] \subseteq [1, dM] \subseteq [1, a+Md] \subseteq [1, W] \subseteq [1, N]$. Furthermore, $|S| = |T| = t_j' = t_j$ and if $y \in P(S)$ then $y = dx$ for some $x \in P(T)$ so $c(y) = c(dx) = c''(x) = j$.

Case 2: $j = i$.

Let $S = \{a+W\} \cup \{dx | x \in T\}$. If $x \in T$ then $dx \leq dM \leq W - a < W + a$ so $|S| = |T| + 1 = t_i' + 1 = t_i$. We have $P(S) = \{dx | x \in P(T)\} \cup \{a+W\} \cup \{a+W+dx | x \in P(T)\}$. If $x \in P(T)$ then $x \leq M$ so $dx \leq a + W + dx \leq W + a + dM \leq 2W = N$. Furthermore, $a + W \leq a + dM + W \leq 2W = N$ so $P(S) \subseteq [1, N]$. For $x \in P(T)$, $c(dx) = c''(x) = j = i$ and $c(W+a+dx) = c'(a+dx) = i$ since $0 \leq x \leq M$. Also $c(a+W) = c'(a) = c'(a+0 \cdot d) = i$. Hence, $c(y) = i$ for every $y \in P(S)$.

This completes the proof of the theorem.

Of course, (I) follows by choosing all the t_i to be equal. (I) was first proved by Rado [19],[18],[20], although independent proofs were given by Sanders [24] as well as Folkman. As mentioned previously, the case $k = 2$ which we denote in the diagram by (SCH) was first proved by Schur [25].

(G). This Schur theorem for groups is most easily stated as follows: Given k and r there is an integer $N(k,r)$ such that if G is a group with $|G| \geq N(k,r)$ and G is arbitrarily r-colored, then G contains k distinct elements $g_1, \ldots, g_k$ all of whose nonempty products have one color.

The proof of (G) given in [11] actually requires a combination of (S) and (I). It is interesting to note that the corresponding result for semigroups does not hold. Simply consider the null semigroup S defined by $ab = 0$ for all $a, b \in S$ and assign to 0 a color not assigned to any other element of S.

(LE). This result, closer to (VdW) than either a true Ramsey or Schur theorem, has the following statement:

(LE) Let $\mathcal{L}$ be a finite system of homogeneous linear equations in the variables $x_1, \ldots, x_t$ with complex coefficients. Suppose for any i, $1 \leq i \leq t$, there is a solution $(\varepsilon_1^{(i)}, \ldots, \varepsilon_t^{(i)})$ to $\mathcal{L}$ with all $\varepsilon_j^{(i)} = 0$ or 1 and $\varepsilon_i^{(i)} = 1$. Then given any r there is an integer $N(\mathcal{L}, r)$ such that if the integers in $[1, N(\mathcal{L}, r)]$ are r-colored then there is a solution $(a_1, \ldots, a_t)$ of $\mathcal{L}$ with all $a_i \in [1, N(\mathcal{L}, r)]$ and all a_i having one color.

(LE) is a special case of some very elegant results (R) of Rado [19],[18], in which necessary and sufficient conditions are derived for the forced existence of monochromatic solutions of systems of linear equations. The derivation of (LE) from (KPS) is fairly straightforward and is given in [11].

It may be pointed [18] out that (LE) leads very directly to (VdW) and (I).

To see this, first consider the system of equations $\mathcal{L}$: $\{x_{i+2}-x_{i+1} = x_{i+1}-x_i : 1 \le i \le n\}$. Since for each x_i there is a (0,1)-solution of $\mathcal{L}$ with $x_i = 1$ (in fact, the single solution (1,1,...,1) satisfies all the conditions) then $\mathcal{L}$ satisfies the hypotheses of (LE). However, the x_i certainly lie in an arithmetic progression so that the conclusion of (LE) implies (VdW).

The derivation of (I) from (LE) is equally simple, starting from the system of equations $\mathcal{L}$: $\left\{\sum_{i\in T} x_i = y_T : \emptyset \neq T \subseteq [1,k]\right\}$. For each x_i, we can construct a (0,1)-solution of $\mathcal{L}$ with $x_i = 1$, $x_j = 0$ for $j \neq i$ and $y_T = 1$ iff $i \in T$. These solutions show that $\mathcal{L}$ satisfies the hypotheses of (LE); the conclusion of (LE) is just (I).

(MM). This result is included to show the application of (KPS) to systems of nonlinear equations. By a system of multigrade equations of order m we mean a system of homogeneous nonlinear equations $\mathcal{N}$ of the form

$$\sum_{k=1}^{t} x_k^i = \sum_{k=1}^{t} y_k^i, \quad i = 0,1,\ldots,m.$$

It is known [14] that if t is sufficiently large as a function of m (e.g., $t \geq 1 + \binom{m+1}{2}$) then the system $\mathfrak{n}$ always has nontrivial integer solutions (i.e., $x_k \neq y_\ell$). In this case, call $\mathfrak{n}$ <u>solvable</u>. (MM) can be stated as follows:

(MM). Given r and a solvable system of multigrade equations $\mathfrak{n}$, there exists an integer $N(\mathfrak{n}, r)$ such that for any r-coloring of the integers in $[1, N(\mathfrak{n}, r)]$, $\mathfrak{n}$ always has a monochromatic solution in $[1, N(\mathfrak{n}, r)]$.

(<u>NC</u>). Let C_n denote the set of vertices of some unit n-cube in euclidean n-space. It is a fact (cf. [11]) that any nonempty subset $X \subseteq C_n$ must span a subspace $\langle X \rangle$ of dimension $\geq \log_2 |X|$. Call X a <u>k-subspace of C_n</u> if $\dim \langle X \rangle = \log_2 |X|$. Let P denote the set of all k-subspaces for all finite k of a fixed countably infinite dimensional unit cube in Hilbert space. P is partially ordered by inclusion; define the rank of a k-subspace to be k. Then the Ramsey theorem $\mathfrak{R}(P)$ is valid for P.

The only proof currently known for (NC) relies on (KPS) (cf. [11]).

<u>OPEN PROBLEMS</u>

The preceding section was meant to give the reader an idea of the main Ramsey and Schur theorems currently known and some of their applications and interrelations. We would like to conclude with a few questions in this area for which very limited information is presently available.

1. It is natural to ask whether or not various <u>infinite</u> analogues of the preceding results are valid. For example, we have seen that an infinite analogue (IS) of (S) is valid.* On the other hand, it is easy to show that (VdW) has no natural infinite generalization. One might ask whether any infinite analogues of (VS) are valid. Similarly, it is not known if there are infinite generalizations of (AP) or (UPS).

The possible infinite analogues to some of the Schur theorems are also quite tantalizing. Although a whole range of infinite cardinals may be used in the various generalizations, even the simplest questions remain unanswered. For example, if the finite subsets of a countably infinite set S are r-colored, must there exist an infinite family $\mathcal{J}$ of disjoint finite subsets of S all of whose (nonempty) unions have one color? What if <u>all</u> subsets of S are colored and elements of $\mathcal{J}$ are allowed to be infinite? It has been noted by Sanders and others that if the requirement of disjointness for elements of $\mathcal{J}$ is dropped then the result is trivially true (by considering an infinite nested family of subsets of S).

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*For a variety of results in this direction, the reader may consult the fundamental papers of Erdös, Hajnal and Rado [3],[4] or the more recent work of Chang [1] and Milner [17].

Similarly one may ask: If the positive integers $\mathbb{Z}^+$ are r-colored, must there exist an infinite subset S all of whose nonempty subset sums have one color?

It is not even known if there must exist an infinite subset S for which all integers in S and all integers in $S \dot{+} S = \{s+s' : s, s' \in S, s \neq s'\}$ have the same color. Of course, the answer is negative if the integers of the form s + s are also required to have the same color: just color the integer k according to the largest power of 2 which divides k. In the positive direction, it is known [6] that for r = 2 there always exists an infinite subset S such that <u>either</u> S and $S \dot{+} S$ all have the same color <u>or</u> S and $S \dot{+} S \dot{+} S \dot{+} S$ all have the same color. Also, it can be shown (using (IS) and (I)) that there always exist infinite subsets $A, B \subseteq \mathbb{Z}^+$ such that all elements of A, B and A + B have one color.

2. Can proofs be given for some of the theorems on the lower portion of Diagram 1 which are essentially simpler than the proofs of the results above them on which they presently depend? For example, can a relatively simple, direct proof be given for (UPS) or (UPI). Can (VdW) be proved just for 2 colors without proving it for an arbitrary number of colors? Are there any obvious arrows missing in Diagram 1, e.g., (S) → (VdW), (VdW) → (S), (AP) → (S′), (UPS) → (VdW), etc.?

Rota has suggested that Ramsey theorems may be valid for many other classes of geometric lattices besides (S), (VS), and (PS). What are some of these? It seems clear that some account must be taken of the "thickness" of the lattice involved. For example, there does not exist a Ramsey theorem for the class of all finite dimensional vector spaces over any finite field partially ordered by inclusion with rank equal to dimension.

3. The results of this paper have been primarily concerned with the existence of Ramsey and Schur theorems for various partially ordered sets P and not with estimates for the ranks involved. Once a particular Ramsey theorem is known to hold, it is of interest to determine bounds on the minimal rank elements for which the theorem is true. Fairly good estimates are available for (S);for some results and further references, see [12] or [2]. For almost all the other cases, the results are considerably more incomplete. Even the bounds for (VdW) are notoriously divergent. The arguments used in [11] to prove (KPS) involve highly recursive applications of the type of arguments which produce the bounds for (VdW).It is therefore not surprising that the best known bounds for (KPS) and its relatives are totally ridiculous,to put it mildly (e.g.,see the estimates for (NC) in [11]).

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