ON SUMS OF FIBONACCI NUMBERS

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For a sequence of integers $S = (s_1, s_2, \cdots)$, we denote by $P(S)$ the set

$$\left\{ \sum_{k=1}^{\infty} \epsilon_k s_k : \epsilon_k = 0 \text{ or } 1, \sum_{k=1}^{\infty} \epsilon_k < \infty \right\}.$$

We say that S is <u>complete</u> if all sufficiently large integers belong to $P(S)$. Conditions under which a sequence S is complete have been studied by a number of authors. These sequences have ranged from the slowly growing sequences of Erdős [3] and Folkman [4] $(s_n = 0(n^2))$, the polynomial and near-polynomial sequences of Roth and Szekeres [9], Graham [5] and Burr [1], to the near-exponential sequences of Cassels [2] $(s_n = 0(\exp(n/\log n)))$ and the exponential sequences of Lekkerkerker [7] and Graham [6] $(s_n = [t\alpha^n])$. In this note, we investigate sequences in which each term is a Fibonacci number, i.e., an integer F_n defined by the linear recurrence

$$F_{n+2} = F_{n+1} + F_n, \qquad n \geq 0,$$

with $F_0 = 0$, $F_1 = 1$.

For a sequence $M = (m_1, m_2, \cdots)$ of nonnegative integers, let S_M denote the nondecreasing sequence which contains precisely m_k entries equal to F_k. It was noted in [7] that for $M = (1, 1, 1, \cdots)$, S_M is complete but the deletion of any two terms of S_M destroys the completeness. Further, it was shown in [1] that for any fixed a, if $M = (a, a, a, \cdots)$ then some finite set of entries can be deleted from S_M so that the resulting sequence is not complete. This result can be strengthened as follows (where τ denotes $(1 + \sqrt{5})/2$).

Theorem 1. If

$$\sum_{k=1}^{\infty} m_k \tau^{-k} < \infty ,$$

then some finite set of entries of S_M can be deleted so that the resulting sequence is not complete.

Proof. The proof uses the ideas of Cassels [2]. Let $\|x\|$ denote $\min |x - n|$ where n ranges over all integers. It is well known that F_n can be explicitly written as

$$F_n = \frac{1}{\sqrt{5}} (\tau^n - (-\tau)^{-n}) .$$

Thus

$$\begin{aligned}
\sum_{s \in S_M} \|s\tau\| &= \sum_{k=1}^{\infty} m_k \|F_k \tau\| \\
&= \sum_{k=1}^{\infty} m_k \|F_k \tau - F_{k+1}\| \\
&= \frac{1}{\sqrt{5}} \sum_{k=1}^{\infty} m_k \left\| \frac{(\tau^2 + 1)}{\tau} (-\tau)^{-k} \right\| \\
&\le \left| \frac{\tau^2 + 1}{\tau\sqrt{5}} \right| \sum_{k=1}^{\infty} m_k \tau^{-k} < \infty
\end{aligned}$$

by the hypothesis of the theorem. Hence, by deleting a sufficiently large initial segment of S_M, we can form a sequence S^*_M for which

$$\sum_{s \in S_M^*} \|s\tau\| < 1/4 \quad .$$

But τ is irrational so that for infinitely many integers m, we have

$$\|m\tau\| > 1/4.$$

The subadditivity of $\| \cdot \|$ shows that such an m cannot belong to $P(S_M^*)$. This proves the theorem.

It follows in particular that if $1 < \theta < \tau$ and $m_k = 0(\theta^k)$ then S_M is not "strongly complete," i.e., the deletion of some finite set of entries from S_M can result in a sequence which is not complete.

In the other direction, however, we have the following result.

Theorem 2. Suppose for some $\epsilon > 0$ and some k_0, $m_k > \epsilon\tau^k$ for $k > k_0$. Then S_M is strongly complete.

Proof. For a fixed integer t, let M' denote the sequence

$$(\underbrace{0,\ 0,\ \cdots,\ 0}_{t},\ m_{t+1},\ m_{t+2},\ \cdots)\ .$$

It is sufficient to show that $S_{M'}$ is complete. We recall the identity

$$F_{n+2k} + F_{n-2k} = L_{2k}F_n\ , \tag{1}$$

where L_r is the sequence of integers defined by $L_{n+2} = L_{n+1} + L_n$, $n \geq 0$, with $L_0 = 2$, $L_1 = 1$. It is easily shown that $F_r \leq \tau^r$ and

$$L_r \geq \frac{1}{2}\,\tau^r$$

for $r \geq 0$. We can assume without loss of generality that $t > k_0$ and $\epsilon\tau^t > 2$. Choose $\ell > 4/\epsilon$ and $n > t + 2\ell$. We can form sums of pairs $F_{n+2k} + F_{n-2k}$ from $S_{M'}$ to get at least $\epsilon\tau^{n-2k}$ copies of $L_{2k}F_n$ (by (1)) for $0 \leq k \leq \ell$. Since $\epsilon\tau^{n-2\ell} > \epsilon\tau^t > 2$ then these sums can be used to form all the multiples uF_n,

$$1 \le u \le \sum_{k=0}^{\ell} \epsilon\tau^{n-2k} L_{2k} .$$

Since

$$L_r \ge \frac{1}{2} \tau^r ,$$

then we have formed all multiples uF_n,

$$1 \le u \le \frac{\epsilon(\ell + 1)}{2} \tau^n .$$

The same argument can be applied to the terms $F_{n+1\pm 2k}$ (which are distinct from the terms previously considered) to form all multiples vF_{n+1},

$$1 \le v \le \frac{\epsilon(\ell + 1)}{2} \tau^{n+1} .$$

Of course, F_n and F_{n+1} are relatively prime so that the set of integers of the form $xF_n + yF_{n+1}$, x and y nonnegative integers, contains all integers $> F_nF_{n+1} - F_n - F_{n+1}$ (cf. [8]). For any integer

$$N_j = F_nF_{n+1} - F_n - F_{n+1} + j, \quad 1 \le j \le F_{n+2} ,$$

the coefficients x_j and y_j in a representation

$$N_j = x_jF_n + y_jF_{n+1}$$

certainly satisfy $x_j \le F_{n+1}$, $y_j \le F_n$. Thus, $x_j, y_j \le \tau^{n+1} < 2\tau^n$. Since u and v can range up to

$$\frac{\epsilon(\ell + 1)}{2} \tau^n > 2\tau^n$$

then by using the multiples of F_n and F_{n+1} we have just considered, we can represent all the N_j, $1 \leq j \leq F_{n+2}$, as elements of $P(S_{M'})$. Finally, since we have used at most $\epsilon\tau^{n-2}$ copies of F_{n+i}, $2 \leq i$, in this process, we still have available at least $\epsilon(\tau^{n+2} - \tau^{n-2}) > 1$ copies of F_{n+i} to use in forming sums in $P(S_{M'})$. By adding sequentially a single copy of F_{n+i}, $i = 2, 3, 4, \cdots$, to the N_j, it is not difficult to see that all integers $\geq N_1$ belong to $P(S_{M'})$. Thus, $S_{M'}$ is complete and the theorem is proved.

It should be pointed out that the condition

$$\sum_{k=1}^{\infty} m_k \tau^{-k} = \infty$$

is not sufficient for the completeness of S_M as can be seen from the example in which

$$m_k = \begin{cases} [\tau^k] & \text{if } k = 2^n \text{ for some } n \\ 0 & \text{otherwise} \end{cases}.$$

However, the proof of Theorem 2 directly applies to show that if m_ϵ/τ^k is monotone and

$$\sum \frac{m_k}{\tau^k} = \infty$$

then S_M is strongly complete.

It would be of interest to investigate refinements of these questions. Of course, similar results and questions arise for other P - V numbers besides τ but we do not pursue these here.

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