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INCREASING PATHS IN EDGE ORDERED GRAPHS

by

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To the memory of A. RÉNYI

Given an undirected graph G having n vertices and q edges let the "ordering" N be a $1-1$ map between the edges of G and the positive integers $\leq q$. A path of length k is a sequence $(e_1, \ldots, e_k)$ of k distinct edges such that e_i and e_{i+1} have a common vertex. A path is simple if the only edges which have a common vertex are of the form e_i, e_{i+1} for some i. An increasing path is one in which $N(e_i) < N(e_j)$ whenever $i < j$.

The following questions have been raised by CHVÁTAL and KOMLÓS [1]: Suppose G is a complete graph K_n so that $q = \binom{n}{2}$. How long an increasing path must exist in G? How long a simple increasing path must exist? If we let $P(N, G)$ and $S(N, G)$ denote the lengths of the longest increasing and simple increasing paths, respectively, in G with the ordering N, then the preceding questions are concerned with

$$f(n) = \min_N P(N, K_n) \quad \text{and} \quad g(n) = \min_N S(N, K_n) .$$

In this note we give a complete answer to the first question and a partial answer to the second. In particular we show that for any edge ordered graph G having n vertices and q edges there is always an increasing path of length at least $2q/n$. From this it will follow that $f(3) = 3$, $f(5) = 5$, $f(n) = n - 1$ for $n \neq 3,5$. The length $g(n)$ of the longest simple increasing path in an edge ordered complete graph K_n has not been determined. We show that $g(n) \geq \frac{1}{2}(\sqrt{4n-3} - 1)$ but this is probably a weak bound, and we obtain a simple construction for which $g(n) < \frac{3n}{4}$. We conjecture that the correct bound is closer to the latter.

The results below are divided into four sections. The first two discuss the lower bounds on $P(N, G)$ and $S(N, K_n)$, respectively, and the latter two deal with the upper bound.

I. Lower bounds on P(N, G)

THEOREM 1. *The longest increasing path in any edge ordered graph G having n vertices and q edges has length at least $2q/n$.*

PROOF. Given an edge ordered graph G on n vertices $v_1, v_2, \ldots, v_n$ we define $p(v_k, G)$ to be the length of the longest increasing path ending at v_k. Suppose the edges of G' in order are $e_1, e_2, \ldots, e_{q'}$, and suppose that G'' has edges, in order, $e_1, \ldots, e_{q'}, e_{q'+1}$ with $e_{q'+1}$ joining v_r to v_s. Then

$$p(v_j, G'') \geqq p(v_j, G') \text{ for any } j,$$

$$p(v_r, G'') \geqq p(v_s, G') + 1$$

and

$$p(v_s, G'') \geqq p(v_r, G') + 1$$

since one can extend the longest increasing path ending at v_s in G' by the edge $e_{q'+1}$ arriving at a path of length $p(v_s, G') + 1$ ending at v_r, etc. Upon adding these relations we obtain

$$\sum_j p(v_j, G'') \geqq \Big(\sum_j p(v_j, G')\Big) + 2. \tag{1}$$

If we start from the empty graph and build up G edge by edge using this argument we obtain the result that if G has q edges then

$$\sum_{j=1}^{n} p(v_j, G) \geqq 2q$$

from which it follows that the average over j of $p(v_j, G)$ is at least $2q/n$. Thus, at least one v_j must have $p(v_j, G)$ as large as $2q/n$ and the theorem is proved.

The argument also indicates the type of ordering which will minimize the maximal $p(v_j, G)$. This will be one which, as far as possible, satisfies (1) with equality and for which all the $p(v_i, G)$ are as equal as possible.

For the complete graph K_n on n vertices, $q = \binom{n}{2}$ so that $\frac{2q}{n} = n - 1$. In Section III we show that for $n \neq 3,5$, one can find an ordering for which $p(v_j, K_n) = n - 1$ for all vertices v_j. This clearly can occur only if the inequalities (1) are always equalities in this ordering.

For $n = 3$, there is essentially only one possible ordering, and for this,

$$\max_j p(v_j, K_3) = 3 = f(3).$$

For $n = 5$, it is possible to exhaust the possible orderings; in each case there is an increasing path of length 5. This result can be verified in a less exhausting manner by an inspection of the sequences $(p(v_1, \bar{G}), p(v_2, \bar{G}), \ldots, p(v_5, \bar{G}))$ with $\bar{G}$ denoting the graph which consists of the first five edges of the edge ordered K_5. If $f(5)$ were equal to 4, there would be an ordering of the edges of K_5 in which all the inequalities (1) were equalities. For such an ordering the sequence above would have to have one of the forms (for some ordering of the vertices)

$$(2, 2, 2, 2, 2),$$

$$(3, 2, 2, 2, 1),$$

$$(3, 3, 2, 1, 1),$$

$$(4, 3, 2, 1, 0).$$

(All other possibilities are easily eliminated.)

One can find an increasing path in K_5 by adding to any increasing path ending at v_j an increasing path in the complement of $\bar{G}$ starting at v_j. The corresponding sequence for the increasing path in the complement of $\bar{G}$ starting at v_j must add to the sequence above to yield (4, 4, 4, 4, 4) if we are to have $f(5) = 4$.

One can easily verify that a sequence (4, 3, 2, 1, 0) can only arise if one vertex is missing in $\bar{G}$. But then every vertex must appear in the complement of $\bar{G}$, so that its sequence contains no 0's. This sequence will not give rise to a (4, 4, 4, 4, 4) sequence. The sequence (2, 2, 2, 2, 2) can only arise if $\bar{G}$ consists of a "4-cycle with a tail". Since the complement of such a graph contains a triangle, such a sequence must force an increasing path of length 5. By enumerating the graphs corresponding to the two remaining sequences (3, 2, 2, 2, 1) and (3, 3, 2, 1, 1), one can easily show that it is not possible to have sequences arising from $\bar{G}$ and its complement which sum to (4, 4, 4, 4, 4). This shows that $f(5) \geqq 5$. On the other hand, the ordering $(e_{12}, e_{34}, e_{51}, e_{23}, e_{45}, e_{13}, e_{24}, e_{35}, e_{41}, e_{52})$ where e_{ij} joins vertices v_i and v_j, is an edge ordering of K_5 in which any increasing path can never contain two consecutive edges in the ordering. This shows that $f(5) = 5$. The upper bound on $f(n)$ for $n \neq 3,5$ will be dealt with in Section III.

II. Lower bounds on S(N, G)

In this section we obtain a lower bound on $g(n)$ of the form $c\sqrt{n}$.

THEOREM 2. *In an edge ordered complete graph K_n there always exists a simple increasing path of length at least* $\frac{1}{2}(\sqrt{4n-3}-1)$.

PROOF. Given an edge ordered graph $\bar{G}$, let $s(v_k, \bar{G})$ denote the length of the longest simple increasing path ending at v_k in $\bar{G}$ and let $t(v_k, \bar{G})$ denote the number of edges e in $\bar{G}$ satisfying:

(a) e joins v_k to v_j, for some j, so that (b) holds;

(b) in the graph $\bar{G}(e)$ consisting of the edges of $\bar{G}$ preceding (and excluding) e in our ordering, all simple increasing paths of length $\geq s(v_k, \bar{G}(e))$ which end at v_k also contain v_j.

If $\bar{G}'$ contains one more edge than $\bar{G}$ has, say, the edge e which joins v_j and v_k, and the edge e follows all the edges of $\bar{G}$ in our ordering, then we have $\bar{G}'(e) = \bar{G}$ and it is not difficult to see that at least one of the following possibilities must occur:

(i) $t(v_j, \bar{G}') \geq t(v_j, \bar{G}) + 1$,

(ii) $t(v_k, \bar{G}') \geq t(v_k, \bar{G}) + 1$,

(iii) $s(v_j, \bar{G}') \geq s(v_k, \bar{G}) + 1, \; s(v_k, \bar{G}') \geq s(v_j, \bar{G}) + 1$.

By adding these relations we obtain

$$\sum_k \big(s(v_k, \bar{G}') + t(v_k, \bar{G}')\big) \geq 1 + \sum_k \big(s(v_k, \bar{G}) + t(v_k, \bar{G})\big)$$

and hence

$$\sum_k \big(s(v_k, K_n) + t(v_k, K_n)\big) \geq \frac{n(n-1)}{2}.$$

On the other hand, as consecutive edges are added to a graph G, the value of $t(v_k, G)$ can increase at most $s(v_k, G) - 1$ times (each time by $+1$) without having the value of $s(v_k, G)$ increase, since by the definition of t, each time $t(v_k, G)$ increases we have added an edge which joins v_k to a vertex v_j lying on all current longest paths to v_k. Thus,

$$t(v_k, K_n) \leq \frac{1}{2} s(v_k, K_n)\big(s(v_k, K_n) - 1\big).$$

Combining these inequalities we obtain

$$\sum_k s(v_k, K_n)\big(s(v_k, K_n) + 1\big) \geq n(n-1)$$

which implies that the average value and hence maximum value of $s(v_k, K_n)$ exceeds $\frac{1}{2}\big(\sqrt{4n-3} - 1\big)$, thus proving the theorem.

With additional effort this estimate can be increased to one of the form $c\sqrt{n}$ for some $c > 1$.

III. Upper bounds for $f(n)$

For a graph G, let $P(G)$ denote $\min_N (P(N, G))$. The following observation asserts that P is subadditive.

FACT.

$$\textit{If } G = \bigcup_k G_k \textit{ then } P(G) \leqq \sum_k P(G_k).$$

PROOF. We can assume without loss of generality that the G_k are disjoint. Let N_k be an edge ordering of G_k such that $P(N_k, G_k) = P(G_k)$. If G_k has q_k edges define an edge ordering N on G by

$$N(e) = \sum_{j<k} q_j + N_k(e) \quad \text{for } e \text{ in } G_k \text{ for all } k.$$

In other words, first all the edges of G_1 are labelled according to N_1, then all the edges of G_2 are labelled in the same order as given by N_2, etc. For this N, we certainly have

$$P(G, N) \leqq \sum_k P(G_k)$$

which completes the proof.

We now apply this result to K_n to show that the bound on $f(n)$ in Section I is exact.

For $n = 2m$, it is easy to decompose K_n into $n - 1$ disjoint *matchings* or 1-factors (i.e., subgraphs consisting of m disjoint edges). But if K is a matching then $P(K) = 1$ so we have

$$f(2m) = P(K_{2m}) \leqq 2m - 1 .$$

For $n = 2m + 1$ we must work a little harder. Suppose G is a graph on k vertices in which each component consists of an even cycle with a (possibly empty) set of simple edges ("tails") at each vertex. Thus, G has k edges and typically looks like the graph in Fig. 1. Let us call such graphs G *admissible*. It is easy to see that if G is admissible then $P(G) = 2$. For, in each component we simply assign the low block of integers to alternate edges in the even cycle, the middle block of integers to the tails and the high block of integers to the remaining edges of the even cycle (cf. Fig. 1). For this ordering N, $P(N, G) = 2$. Thus, to show that the bound of Section I is exact, it suffices

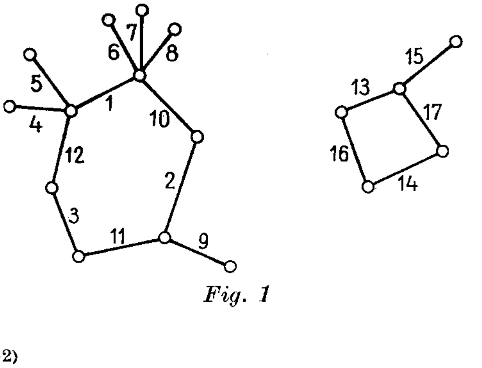

Fig. 1

to show that for $n = 2m + 1 \geqq 7$, K_n can be decomposed into m admissible subgraphs.

We first exhibit some special decompositions for K_7, K_9, K_{11} in Fig. 2. These decompositions each have the properties:

(i) K_n is the edge disjoint union of the $G_{n,i}$, $1 \leqq i \leqq m$.

(ii) Each $G_{n,i}$ is admissible and has the disjoint union of A_n, B_n and $\{g_n\}$ as its set of vertices.

(iii) $|A_n| = |B_n| = m$.

(iv) $\alpha_{n,i}$ and $\beta_{n,i}$ belong to even cycles in $G_{n,i}$.

(v) $\bigcup_{1 \leqq i \leqq m} \{\alpha_{n,i}\} = A_n$, $\bigcup_{1 \leqq i \leqq m} \{\beta_{n,i}\} = B_n$.

Let us call such a decomposition *special*.

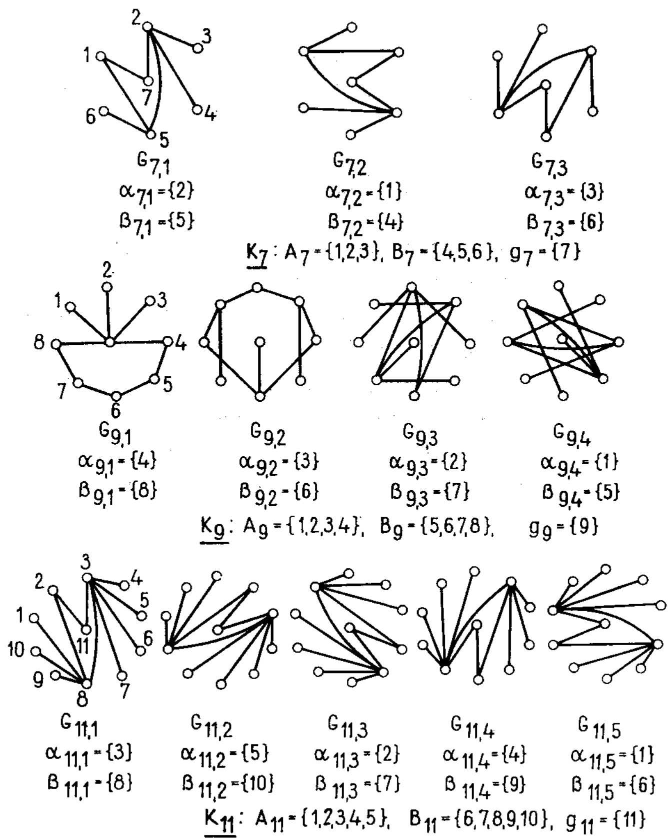

Fig. 2

We next show that if K_{2m+1} and $K_{2m'+1}$ both have special decompositions then so does $K_{2m+2m'+1}$. We imagine starting with disjoint copies of K_{2m+1} and $K_{2m'+1}$ and their respective special decompositions and we identify the vertex g_{2m+1} in K_{2m+1} with the vertex $g_{2m'+1}$ in $K_{2m'+1}$, giving us a total of $2m + 2m' + 1$ distinct vertices. We define:

$$A_{2m+2m'+1} = A_{2m+1} \cup A_{2m'+1}, \quad B_{2m+2m'+1} = B_{2m+1} \cup B_{2m'+1}$$

$$g_{2m+2m'+1} = g_{2m+1} = g_{2m'+1}$$

and

(a) For $1 \leq k \leq m$, $\alpha_{2m+2m'+1,k} = \alpha_{2m+1,k}$, $\beta_{2m+2m'+1,k} = \beta_{2m+1,k}$ and $\{u, v\}$ is an edge of $G_{2m+2m'+1,k}$ iff $\{u, v\}$ is an edge of $G_{2m+1,k}$ or $u = \alpha_{2m+1,k}$, $v \in A_{2m'+1}$ or $u = \beta_{2m+1,k}$, $v \in B_{2m'+1}$;

(b) For $m + 1 \leq k \leq m + m'$, $\alpha_{2m'+2m+1,k} = \alpha_{2m'+1,k}$, $\beta_{2m+2m'+1,k} = \beta_{2m'+1,k}$ and $\{u, v\}$ is an edge of $G_{2m+2m'+1,k}$ iff $\{u, v\}$ is an edge of $G_{2m'+1,k}$ or $u = \alpha_{2m'+1,k}$, $v \in B_{2m+1}$ or $u = \beta_{2m'+1,k}$, $v \in A_{2m+1}$.

It is straightforward to verify that this decomposition of $K_{2m+2m'+1}$ does satisfy (i)—(v). Thus, since we can choose $m' = 3$ then we see that if K_{2m+1} has a special decomposition then so does K_{2m+7}. Since K_7, K_9, and K_{11} have special decompositions then K_{2m+1} also has, for all $m \geq 3$. This shows in particular that each K_{2m+1}, $m \geq 3$, can be partitioned into m admissible subgraphs and therefore $P(K_{2m+1}) \leq 2m$, $m \geq 3$.

We can combine all the preceding results to give

THEOREM 3.

$$f(n) = P(K_n) = \begin{cases} n - 1 & \text{for } n \neq 3,5 \\ n & \text{otherwise.} \end{cases}$$

IV. Upper bounds for g(n)

In contrast to the sharp results we have for $f(n)$, the corresponding bounds known for $g(n)$ are much less precise. The current best upper bound on $g(n)$, namely, $g(n) < \frac{3n}{4}$, is obtained in the following way.

Partition the vertices of K_{4m+k}, $0 \leq k \leq 3$, into 4 subsets S_i, with $|S_i| = m + 1$, $1 \leq i \leq k$, and $|S_i| = m$, $k < i \leq 4$. Label all the edges of K_{4m+k} in the following order:

(i) First the edges in S_1, S_2, S_3 and S_4;

(ii) Next, the edges between S_1 and S_2, then those between S_3 and S_4;

(iii) Next, the edges between S_1 and S_3, then those between S_2 and S_4;

(iv) Finally, the edges between S_1 and S_4, and last, those between S_2 and S_3.

It is easily seen that no simple increasing path can contain vertices in all four of the S_i. Thus,

$$P(K_{4m+k}) \leq 3m + k - 1, \qquad 0 \leq k \leq 3\ ,$$

and

$$g(n) = P(K_n) < \frac{3n}{4}$$

as asserted.

REFERENCE

[1] V. Chvátal and J. Komlós, Some combinatorial theorems on monotonicity, *Canad. Math. Bull.* **14** (1971).

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