SOME RECENT DEVELOPMENTS IN RAMSEY THEORY

R.L. GRAHAM

Bell Laboratories, Murray Hill, New Jersey 07974, USA

B.L. ROTHSCHILD

University of California, Los Angeles, Cal. 90024, USA and Bell Laboratories, Murray Hill, New Jersey 07974, USA

Recently a number of striking new results have been proved in an area becoming known as RAMSEY THEORY. It is our purpose here to describe some of these. Ramsey Theory is a part of combinatorial mathematics dealing with assertions of a certain type, which we will indicate below. Among the earliest theorems of this type are RAMSEY's theorem, of course, VAN DER WAERDEN's theorem on arithmetic progressions and SCHUR's theorem on solutions of $x+y = z$.

To make our task easier, we will introduce the "arrow notation" of ERDÖS and RADO. This was originally used for generalizations of Ramsey's Theorem to infinite cardinals, but can be easily adapted to other cases as well. The meaning of the arrow notation will become clear by its use in the examples throughout this paper.

As our first example, consider:

$$n \underset{k}{\rightarrow} (\ell_1,\ldots,\ell_r) .$$

This expression is just an abbreviation for the following assertion: if the k-element subsets of an n-element set are partitioned into r classes, then for some i there is an ℓ_i-element subset L_i of the n-element set such that all the k-element subsets of L_i are in the i-th class.

<u>THEOREM</u>. (RAMSEY). *For all positive integers* $k,r,\ell_1,\ldots,\ell_r$, *there exists an* $N = N(k,r,\ell_1,\ldots,\ell_r)$ *such that if* $n \geq N$, *then* $n \underset{k}{\rightarrow} (\ell_1,\ldots,\ell_r)$.

In fact, RAMSEY considered only the case where all the ℓ_i are equal.

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He also proved $\aleph_0 \xrightarrow[k]{} (\underbrace{\aleph_0, \ldots, \aleph_0}_{r})$ which actually is stronger than the finite theorem above. The consideration of such statements with large cardinals or ordinals is a subject in itself and will not be discussed here. For the large cardinals the subject is fairly complete and will be covered in a forthcoming book of ERDÖS, HAJNAL & RADO. For ordinals, the theory is developing rapidly, although there are still many open questions. To give the flavor of a result of this type, we mention one of the most interesting recent ones.

THEOREM. (CHANG, LARSEN, MILNER). $\omega^{\omega} \xrightarrow[2]{} (\omega^{\omega}, k)$

This theorem asserts that if the pairs (i.e., 2-element subsets) of a set of order type ω^{ω} are partitioned into two classes, then either the first class contains all the pairs of a subset with induced order type ω^{ω}, or the second class contains all the pairs of some k-element subset.

This last example illustrates the arrow notation in a case where we deal with sets with structure (here the structure is that of order).

In general, in a Ramsey Theorem an assertion of the form $A \xrightarrow[B]{} (C_1, \ldots, C_r)$, where the symbols A, B and C_i denote objects with a certain structure. For example, as above, they could be sets or sets with order. Other examples include graphs, finite vector spaces, sets containing solutions to systems of linear equations, Boolean algebras and partitions of finite sets.

In the remainder of the paper, we will consider six examples of Ramsey theorems. The first two concern graphs and are due to W. DEUBER and to J. NEŠETŘIL & V. RÖDL. The next three concern systems of linear equations and their solution sets. These are results of N. HINDMAN, E. SZEMEREDI and W. DEUBER. Finally, we will discuss some results of K. LEEB on abstract categories which are *"Ramsey"*.

GRAPHS

Recalling the previous statement of Ramsey's Theorem, we see that the first non-trivial case is

$$6 \xrightarrow[2]{} (3,3) \ .$$

This can be restated as follows: if the edges of the complete graph K_6 on six vertices are 2-colored arbitrarily, then some monochromatic triangle K_3 must be formed. This graphical form leads to several general considerations. The most natural of these, an immediate consequence of Ramsey's Theorem (with $k = 2$), is simply:

For every finite graph H, *there is a finite graph* G *such that* $G \underset{2}{\rightarrow} (H,H)$.

Here, the arrow notation means that if the *edges* of G (represented by the 2 below the arrow) are 2-colored arbitrarily, then G will contain a monochromatic subgraph isomorphic to H.

It would be stronger to require that the monochromatic subgraph above be an *induced* subgraph of G. We could write $G \underset{2}{\rightarrow} (H,H)$ also in this case, provided we understand that we mean induced subgraphs here. Actually, to be rigorous, we should use a "different" kind of arrow for each different meaning. The proper setting for this is in terms of category theory as originally indicated by LEEB. We will elaborate on this when we discuss LEEB's recent results at the end of this paper.

We now turn our attention to the first result, which concerns *induced* subgraphs of graphs.

THEOREM. (DEUBER [2]). *For every finite graph* H, *there exists a finite graph* G *such that* $G \underset{2}{\rightarrow} (H,H)$.

SKETCH OF PROOF. What DEUBER actually proves is the equivalent but more convenient statement: for every choice of finite graphs G and H there exists a finite graph K such that $K \underset{2}{\rightarrow} (G,H)$. The proof is by induction on $|G|+|H|$ where $|G|$ denotes the number of vertices of G. The small cases are trivial. Let g be a vertex of G, $\bar{G} = G-\{g\}$, and let S be the subset of G to which g is connected. Also, let h in H, $\bar{H}$ and T be defined similarly.

By induction we can find G^* and H^* such that $G^* \underset{2}{\rightarrow} (\bar{G},H)$ and $H^* \underset{2}{\rightarrow} (G,\bar{H})$. We now form a large graph K as follows: Start with G^*. Let $\bar{G}_1,\ldots,\bar{G}_m$ be all the occurrences of $\bar{G}$ as an induced subgraph of G^* and let $S_1,\ldots,S_m$ be the corresponding subsets S (there may be more than one choice for an S_i; any one is allowed). Now replace each vertex of $S = S_1 \cup \ldots \cup S_m = \{x_1,\ldots,x_\ell\}$ by a complete copy of H^*, with the copy of H^* replacing x_i denoted by H_i^*. Connect a vertex of H_i^* to a vertex of H_j^* iff x_i and x_j are connected in G^*. Also, if some vertex v is not in S, connect v to all the vertices of H_i^* iff v and x_i are connected in G^*. Thus, we have essentially "exploded" some of the vertices of G^* into H^*'s.

Suppose, in the simplest case, that all the S_i are disjoint. Let $\bar{H}_1,\ldots,\bar{H}_n$ be the occurrences of $\bar{H}$ in H^* and let $T_1,\ldots,T_n$ denote the corresponding subsets T. For each fixed S_i, consider the associated H^*'s and choose one T_j from each H^*. For each i and such choice of T_j's, we introduce a new vertex connected exactly to these T_j's. Hence, if $|S_i| = k$, then for this i we have added n^k new vertices. Since we have m disjoint S_i, then there are altogether mn^k new vertices. This completes the definition of K.

Suppose now the edges of K are 2-colored, say, using the colors red and blue. By the construction of H^*, each H^* in K has either a red copy of G or a blue copy of $\overline{H}$. If the first alternative holds, then we are done. So assume each H^* in K contains a blue copy of $\overline{H}$. Let $y_1,\ldots,y_m$ be the new vertices corresponding to the subsets T_j for these copies of $\overline{H}$ (i.e., one y_i for each S_i). If any of the y_i are connected to any of the T_j by all blue edges, we are done since in this case we have a blue copy of H. Thus, we may assume that each y_i is connected by a red edge to some vertex $T \in T_j$ for each T_j to which it is connected. Let y_i be connected by red edges to $t_{i1},\ldots,t_{iw}$. Consider the graph $\tilde{G}$ obtained from K by deleting all the vertices of all the copies of H^* except for the t_{ij}, and deleting all the new vertices except $y_1,\ldots,y_m$. By construction, $\tilde{G}$ is isomorphic to G^* together with the y_i. Also, it is an induced subgraph of K. Since each y_i is connected to the corresponding S_i by only red edges, we are done. For either $G^* \subseteq \tilde{G}$ contains a blue copy of H or, it contains a red copy of $\overline{G}$, say $\overline{G}_i$, which together with y_i forms a red copy of G. This completes the argument for the case that the S_i are disjoint.

The only obstruction preventing this from being completely general is that it usually happens that for some a and b, $S_a \cap S_b \neq \emptyset$ in G. This in turn would prevent us from choosing the same t_{ij} for both S_a and S_b when necessary. To get around this, we add another step to the construction. Namely, after replacing the vertices of S by copies of H^*, we take those in the $S_a \cap S_b$ and replace each vertex of the H^* itself by a copy of H^*, connecting it up in the same way as before. We can then be certain of obtaining a vertex connected by only red edges to some copy of H^*, and we can proceed essentially in the same way as before. □

Of course, the graphs K resulting from this construction are usually much larger than are actually required. For example, the graph K constructed this way for the assertion $K \underset{2}{\rightarrow} (K_3,K_3)$ is $K = K_{81}$. Note also the high clique number K_{81} has relative to that of K_3.

F. GALVIN had asked if for each finite graph H with clique number $cl(H) = k$ (where $cl(H) = \max\{n \mid K_n \text{ is a subgraph of } H\}$), there is a graph G also having $cl(G) = k$ such that $G \underset{2}{\rightarrow} (H,H)$. As above, we consider induced subgraphs here. This question has been very recently answered in the affirmative by J. NEŠETŘIL & V. RÖDL. One sees easily that this implies DEUBER's result.

FOLKMAN, in response to a question of ERDÖS and HAJNAL, had earlier shown that there exists a graph G with cl(G) = k such that $G \underset{2}{\rightarrow} (K_k, K_k)$. FOLKMAN also proved that for any G and H, there exists a K with cl(K) = = max{cl(G), cl(H)} such that $K \underset{1}{\rightarrow} (G,H)$ (where the 1 below the arrow indicates that we are coloring the vertices of K instead of the edges). In fact, NEŠETŘIL & RÖDL also make use of this theorem.

The second result we discuss is the following:

<u>THEOREM</u>. (NEŠETŘIL & RÖDL). *For every finite graph* H *there exists a finite graph* G *such that* $G \underset{2}{\rightarrow} (H,H)$ *and* cl(G) = cl(H).

<u>SKETCH OF PROOF</u>. The proof uses the ingenious idea of letting the vertices of G be subsets of a large set. By appropriately defining when edges occur between them, and applying Ramsey's Theorem to certain subsets, a large subset is obtained with the vertices and edges determined by it being very well behaved.

We will make some definitions first, and then indicate somewhat how the proof goes, especially for the case of cl(H) = 2, which is considerably simpler and more direct than the general case. We begin with the definition of the graphs (n,T,p).

Let A,B be two p-subsets of $[1,n] = \{1,2,\dots,n\}$. The *type* (or *p-type*) t(A,B) of A and B is the pattern of their relative order, defined as follows: List the elements of $A \cup B$ in increasing order assuming $\min\{A-B\} < \min\{B-A\}$, say $x_1, x_2, \dots, x_\ell$, $\ell \le 2p$. If $x_i \in A \cap B$ replace it by two copies of itself. The new list thus obtained, say $y_1, y_2, \dots, y_{2p}$, is of length 2p. The type t(A,B) is then defined to be the sequence $(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{2p})$, where $\bar{y}_i = 2$ if $y_i \in A \cap B$, $\bar{y}_i = 0$ if $y_i \in A-B$, and $\bar{y}_i = 1$ if $y_i \in B-A$. We let t(B,A) = t(A,B).

Let T be a set of p-types. The graph (n,T,p) is defined by having as vertices all $\binom{n}{p}$ p-subsets of [1,n], and as edges, all pairs A,B of p-subsets with $t(A,B) \in T$. We define the clique number of T by cl(T) = $= \sup_n cl((n,T,p))$. (Not all T have finite clique number, e.g., $\{(0,1)\} = T$, although some do.)

The beautiful construction of (n,T,p) has the property that for large n it is extremely rich in induced subgraphs (m,T,p), for m < n. This enables us to use Ramsey's Theorem ultimately to obtain very well behaved subgraphs.

The first result in the general case is to show that for each H there exist p and T, so that H is an induced subgraph of (n,T,p) for all large n, and with cl(T) = cl(H). T and p are defined inductively, in general, and are quite complicated. However, for cl(H) = 2, we can describe T much more simply and, in fact, we can assert even more. Namely, for each H, let its vertices be ordered arbitrarily, say, $x_1, x_2, \ldots, x_k$. Then there is a mapping Φ: H $\to$ (n,T,p) for a suitable n,T,p such that cl(T) = 2, and Φ maps H isomorphically into an induced subgraph of (n,T,p) with $t(\Phi(x_i),\Phi(x_j))$ depending only on j, if i < j. In the general case cl(H) = k, a similar result holds, but the proof is much more complicated. For the remainder of the discussion, we restrict ourselves to cl(H) = 2. The mapping Φ is defined inductively. T is the set of all types starting with some 0's, two 2's, then 0's and 1's only, e.g., (0,0,0,2,2,1,1,0,1,0,1,1). It is easy to see that cl(T) = 2.

Now suppose for large N that the edges of (N,T,p) are 2-colored. For each (2p-1)-subset S of [1,N] there are $\frac{(2p-1)}{2}\binom{2p-2}{p-1}$ pairs of p-subsets A,B with $A \cup B = S$. Of those pairs, some number m have their type in T. If we list these in some canonical order, say lexicographically, then we get for each $A \cup B$ a list of m types, corresponding to m edges, and thus m colors. But this produces a 2^m-coloring of the (2p-1)-subsets of [1,N]. Thus, for any n, if N is large enough, Ramsey's Theorem implies that there is a subgraph (n,T,p) of (N,T,p) with all edges of a given type having the same color.

Let H be an arbitrary graph with cl(H) = 2, and let G^* be such that $G^* \underset{1}{\to} (H,H)$, which exists by FOLKMAN's result. Letting Φ be as above, we have $\Phi(G^*) \subseteq (n,T,p) \subseteq (N,T,p)$. Each vertex x_j of G^* is associated with a single type $t(\Phi(x_i),\Phi(x_j))$ for i < j, and thus with a single color. By choice of G^*, then, we obtain a subgraph H all of whose vertices have the same color. But by the definition of this coloring, all edges of H have the same color. This completes the case cl(H) = 2, since by letting G = (N,T,p) we have $G \underset{2}{\to} (H,H)$. As previously remarked, the proof for the general case cl(H) = k is similar in spirit but with somewhat more complicated details. □

LINEAR EQUATIONS

Let $L = L(x_1,\ldots,x_n)$ denote a finite system of homogeneous linear equations in the variables $x_1,\ldots,x_n$ with integer coefficients. For a set S of integers, we write $S \to (\underbrace{L,\ldots,L}_{r})$, if L always has a monochromatic solution for any r-coloring of S. A system L is said to be *regular*, if, for all r, $\mathbb{P} \to \underbrace{(L,\ldots,L)}_{r}$, where $\mathbb{P}$ denotes the set of positive integers.

R. RADO has characterized all regular L by generalizing the properties of the two best known examples. These are, respectively, L_2: x+y = z and $L(k)$: $x_1-x_2 = x_2-x_3 = \ldots = x_{k-1}-x_k$. That L_2 is regular is SCHUR's theorem. Of course, the regularity of $L(k)$ is trivial (by choosing all the x_i equal). However, if we rule out this possibility, then a solution of $L(k)$ determines an arithmetic progression of length k. This restricted regularity of $L(k)$ for all k is just VAN DER WAERDEN's well-known theorem.

Unfortunately, however, this surprising result still does not specify *which* color these progressions have. It was conjectured some 40 years ago by ERDÖS and TURÁN that a solution must always occur in the most frequently occurring color. More precisely, they conjectured that if R is an infinite sequence of integers with positive upper density, i.e.,

$$(*) \qquad \overline{\lim_{n\to\infty}} \frac{|R \cap [1,n]|}{n} > 0 ,$$

then R contains arbitrarily long arithmetic progressions. No progress was made on this problem until 1954 when K.F. ROTH showed that if R satisfies (*), then R at least contains a *three*-term arithmetic progression. In fact, he showed more, namely, that for some c > 0, if $|R \cap [1,n]| > \frac{cn}{\log\log n}$ then R must contain a three-term arithmetic progression. The next significant step was not made until 1967 when SZEMERÉDI proved that (*) implies that R contains a *four*-term progression. However, SZEMERÉDI's most recent result, which must be considered an achievement of the first magnitude, finally settles the original conjecture of ERDÖS and TURÁN in the affirmative.

THEOREM. (SZEMERÉDI). (*) *implies* R *contains arbitrarily long arithmetic progressions.*

SKETCH OF SKETCH OF PROOF. SZEMERÉDI's proof is completely combinatorial in nature and is based on a lemma on bipartite graphs which is of considerable importance in its own right. We shall give a very brief discussion of the flavor of the proof (which runs just under 100 pages in length), although we can only hint at the extreme ingenuity used in the proof itself.

Let G denote a bipartite graph with vertex sets A and B. We call G regular if all vertices in A have the same degree and all vertices in B have the same degree. We would like to assert that every sufficiently large bipartite graph can be decomposed into a relatively small number of regular bipartite subgraphs, but unfortunately this is not true. However, it is true if the subgraphs are only required to be "approximately" regular and if we are allowed to ignore a small fraction of the vertices in A and B. More precisely, for $X \subseteq A$, $Y \subseteq B$, let $k(X,Y)$ denote the number of edges in the graph induced by the vertex sets X and Y and let $\beta(X,Y)$ denote $\frac{k(X,Y)}{|X||Y|}$, the density of edges in this induced subgraph. Then SZEMERÉDI proves the following:

<u>LEMMA</u>. *For all* $\varepsilon_1,\varepsilon_2,\delta,\rho,\sigma$ *strictly between* 0 *and* 1, *there exist integers* m_0,n_0,M,N *such that for all bipartite graphs* G *with* $|A| = m > M$, $|B| = n > N$ *there exist disjoint* $C_i \subseteq A$, $0 \le i < m_0$, *and for each* $i < m_0$, *disjoint* $C_{i,j}$, $j < n_0$, *such that:*

(a) $|A - \bigcup_{i<m_0} C_i| < \rho m$, $|B - \bigcup_{j<n_0} C_{i,j}| < \sigma n$ *for any* $i < m_0$;

(b) *for all* $i < m_0$, $j < n_0$, $S \subseteq C_i$, $T \subseteq C_{i,j}$, *with* $|S| > \varepsilon_1|C_i|$, $|T| > \varepsilon_2|C_{i,j}|$, *we have* $\beta(S,T) \ge \beta(C_i,C_{i,j})-\delta$;

(c) *for all* $i < m_0$, $j < n_0$ *and* $x \in C_i$, $\beta(\{x\},C_{i,j}) \le \beta(C_i,C_{i,j})+\delta$.

Condition (a) says that we have not omitted too many vertices in the decomposition. Conditions (b) and (c) express the approximate regularity of the subgraphs induced by the vertex sets C_i and $C_{i,j}$.

The basic objects dealt with in the proof are not just arithmetic progressions, but more general structures known as *configurations*. A 1-configuration is just a finite arithmetic progression; an m-configuration is a finite arithmetic progression of (m-1)-configurations.

Let R be an arbitrary fixed set of integers having positive upper density. The basic idea is to show inductively that there exist very long m-configurations which have an extremely restricted manner in which they intersect R. This is done by recursively defining certain special classes of higher order configurations in terms of rather well-behaved progressions of lower order configurations. Essentially, by showing that there exist extremely long configurations of some order which are moderately "regular", one can deduce the existence of configurations of a higher order which are even more "regular". This in turn is done by forming bipartite graphs based on the intersection patterns of the configurations with R and applying the decomposition lemma. Needless to say, the subtlety of the ideas used can only be appreciated by reading the actual proof. □

Turning our attention back to SCHUR's system L_2, we can generalize this to the system L_k defined as follows: for the variables x_s and y_S, L_k consists of all equations of the form $\sum_{s\in S} x_s = y_S$ where S ranges over all non-empty subsets of [1,k]. RADO's results imply that for all k and r,

$$\mathbb{P} \rightarrow (\underbrace{L_k,\ldots,L_k}_{r})$$

It is natural to ask what happens for the system

$$L_\infty = \{\textstyle\sum_{s\in S} x_s = y_S \mid S \subseteq \mathbb{P},\ 1 \le |S| < \infty\}.$$

N. HINDMAN's remarkable theorem answers this question.

THEOREM. (HINDMAN). *For all* r, $\mathbb{P} \rightarrow (\underbrace{L_\infty,\ldots,L_\infty}_{r})$.

SKETCH OF PROOF. In the case L_k it is even true that for each r there is an $N = N(k,r)$ such that $N \rightarrow (\underbrace{L_k,\ldots,L_k}_{r})$. In other words, no matter which r-coloring we have, values $x_1,\ldots,x_k$ can be chosen from [1,N] so that all the sums $\sum_{s\in S} x_s$ have the same color. For a fixed r-coloring of $\mathbb{P}$ restricted to [1,N(k,r)] it was not known whether upper bounds for the x_i existed independent of k. The existence of such bounds would allow HINDMAN's theorem to be obtained directly by a "compactness" argument.

What HINDMAN proves is that for each coloring π of $\mathbb{P}$ with a finite number of colors, there is a function $f_\pi: \mathbb{P} \rightarrow \mathbb{P}$ such that for each m, $0 < m < \infty$, there is a set $x_1,\ldots,x_m$ with all its finite sums the same color, and in addition, such that $x_i \le f_\pi(i)$ for $1 \le i \le m$. That is, we get monochromatic solutions to L_m for arbitrarily large m, where the sizes of the variables x_i are bounded above independently of k but depending on the coloring π (of all of $\mathbb{P}$).

We can illustrate several of the ideas of the proof, but we need some notation first. Let π be a finite coloring of $\mathbb{P}$, say $\mathbb{P} = A_1\cup\ldots\cup A_r$. For $1 \le k \le n$, we define

$$F_\pi(k,n) = \{x \in \mathbb{P} \mid x \ge n \text{ and } \exists i \text{ such that } k,x,x+k \in A_i,\ x \in F_\pi(j,n),\ j < k\}.$$

The $F_\pi(k,n)$ are sets which can be translated by k without changing color. If $x_1,x_2,\ldots$ is a sequence of integers, let $S(x_i)$ be the set of finite sums of the x_i.

The core of HINDMAN's proof is an "exceedingly technical" and quite clever argument, which establishes that for each π there is an infinite sequence $x_1, x_2, \ldots$ and $n \in \mathbb{P}$ such that

$$S(x_i) \subseteq \bigcup_{k=1}^{n-1} F_\pi(k,n).$$

To manipulate sequences and sums conveniently, it would be nice to know that the numbers in the sequences were representable to base 2 in the following manner, e.g.,

$$\begin{aligned} x_1 &= 10110111 \\ x_2 &= 110000100000000 \\ x_3 &= 1100000000000000000 \\ &\vdots \end{aligned}$$

That is, the support of x_j should be all beyond the support of x_{j-1} for each j. Formally, if $2^{s-1} \le x_{j-1}$, then $2^s | x_j$. Such a sequence will be called a *good sequence*. Now for every sequence $x_1, x_2, \ldots$ there is a good sequence (not necessarily a subsequence of the x_i's) $y_1, y_2, y_3, \ldots$ with $S(y_i) \subseteq S(x_i)$. This follows from a compactness argument again.

Hence we basically need to deal only with good sequences. The nice property of these is that if $X = \{x_1, x_2, \ldots\}$ is a good sequence, there is a bijection τ_X: $S(x_i) \to \mathbb{P}$ which preserves sums, namely,

$$\tau_X\Bigl(\sum_{s\in S} x_i\Bigr) = \sum_{s\in S} 2^{s-1}.$$

That is, each block of support corresponds under τ to a single binary place.

We use this fact crucially in the following construction. Suppose π is a coloring, and $x_{\pi 1}, x_{\pi 2}, x_{\pi 3}, \ldots$ is a good sequence with

$$S(x_{\pi i}) = \bigcup_{k=1}^{n(\pi)-1} F_\pi(k, n(\pi)).$$

Then using the map τ_π determined by this sequence, we can get a new coloring π' of $\mathbb{P}$ by letting two numbers have the same π'-color iff their images under τ_π^{-1} are in the same $F_\pi(k,n(\pi))$. This is an $(n(\pi)-1)$-coloring.

Suppose for *all* π we have defined $f_\pi(i)$ for $i \le \ell$ so that arbitrarily long finite good sequences have monochromatic sums and the i-th term is at most $f_\pi(i)$, $i \le \ell$, where we take $f_\pi(1) = n(\pi)-1$ (which works for $\ell = 1$ by the definition of $n(\pi)$). Then consider such a sequence for the coloring π' associated as above with π. Taking τ_π^{-1} of this sequence, we get a similar sequence which is constrained by the definition of π' to have its first ℓ terms respectively less than $\tau_\pi^{-1}(f_{\pi'}(i))$, $i \leq \ell$, and all greater than $n(\pi)-1$. Further, they must all have the same π-color, and for some common $k \leq n(\pi)-1$, adding k does not change this color. Then adjoining k as a first term gives us a new sequence with the first term not exceeding $f_\pi(1)$. Also, if we let $f_\pi(j) = \tau_\pi^{-1}(f_\pi(j-1))$, we have the j-th term not exceeding $f_\pi(i)$ for $j \leq \ell+1$.

We have thus constructed, simultaneously for all π, the bounds $f_\pi(i)$. What we have shown, then, is that for each $\pi = A_1 \cup A_2 \cup \ldots \cup A_r$, and each k, there is a sequence $x_1,\ldots,x_k$ with all its sums in some $A_{i(k)}$ and $x_i \leq$ $\leq f_\pi(i)$, $1 \leq i \leq k$. As we noted above, a compactness argument now completes the proof. □

We remark that because the supports of the x_i in a good sequence are disjoint, we can interpret the x_i as disjoint subsets of $\mathbb{P}$ and their sums as disjoint unions. Thus, we obtain: *for every r-coloring of the finite subsets of* $\mathbb{P}$, *there exists an infinite sequence of finite disjoint sets* $A_1,A_2,A_3,\ldots$ *such that all the finite unions have the same color.*

The last of the results on equations is that of DEUBER, who settles a conjecture RADO raised in his original work. We recall that a system L of homogeneous linear equations is called regular if for any r, $\mathbb{P} \to (\underbrace{L,\ldots,L}_{r})$. RADO defined a *set* $S \subseteq \mathbb{P}$ to be *regular* if for every regular system L and any r, $S \to (\underbrace{L,\ldots,L}_{r})$. What RADO conjectured and what DEUBER proves is the following:

<u>THEOREM</u>. (DEUBER). *If* $S \subseteq \mathbb{P}$ *is regular and* $S = A \cup B$, *then either* A *or* B *is regular.*

<u>SKETCH OF PROOF</u>. The main idea of DEUBER's proof is to define certain sets, called (m,p,c)-sets, and to characterize regular sets in terms of (m,p,c)-sets. He then proves a finite RAMSEY theorem for these sets. Finally, by considering the nice structure of (m,p,c)-sets, he uses a compactness argument to establish the desired result.

We define (m,p,c)-sets below. However, we can describe them informally as a kind of ℓ-dimensional array of numbers (actually, certain subsets of these).

DEFINITION. For m,p,c positive integers, $p \geq c$, an *(m,p,c)-set* A is a set for which there exist m positive integers $a_1, a_2, \dots, a_m$, such that $A = \{ \sum_{i=1}^{m} \lambda_i a_i \mid |\lambda_i| \leq p$, and the first non-zero coefficient λ_i has the value c$\}$.

Now using RADO's characterization of regular systems of equations, we can show the following two facts:

(a) for every regular system L there exist m,p,c such that every (m,p,c)-set contains a solution to L;

(b) for all m,p,c there is a regular system L such that every solution set for L contains an (m,p,c)-set.

As an example, consider the single equation $x+y = z$. Then a solution is any set of the form a_1, a_2, a_1+a_2, which is certainly contained in the (2,1,1)-set generated by a_1 and a_2. On the other hand, the equations $x+y = z_1$, $x-y = z_2$, have solutions exactly of the form $x,y,z_1,z_2 = a_1,a_2,a_1+a_2,a_1-a_2$, a (2,1,1)-set. These examples avoid $c \neq 1$, which can arise when the coefficients are more complicated.

By (a) and (b) we see that a regular set is any set containing (m,p,c)-sets for all m,p,c.

Suppose now that we know the following: for each (m,p,c) there is an (n,q,d) such that $(n,q,d) \to ((m,p,c),(m,p,c))$. That is, if the elements of any (n,q,d)-set are 2-colored, then there must be a monochromatic (m,p,c)-set. Thus for S regular, and $S = A \cup B$, either A or B must contain "arbitrarily large" (m,p,c)-sets and hence, by what we have noted, either A or B is regular. The main part of DEUBER's proof is concerned then with establishing the Ramsey property $(n,q,d) \to ((m,p,c),(m,p,c))$.

This result is similar to one of GALLAI concerning "n-dimensional arrays". For our purposes, we may consider an n-dimensional array as a set of the form

$$X_{n,p} = \{a_0 + \sum_{i=1}^{n} \lambda_i a_i \mid |\lambda_i| \leq p\}.$$

For these we have that for n, p and r there is an N such that $X_{N,p} \to (\underbrace{X_{n,p}, \dots, X_{n,p}}_{r})$.

However, this isn't quite good enough for our purposes, since an (m,p,c)-set will contain sums of the form $ca_i + \sum_{j>i} \lambda_j a_j$ along with certain differences as well (e.g., a_1, a_1+a_2 and a_2 in the example above) while

$X_{N,p}$ may not contain any of its differences. To handle this problem we proceed iteratively.

First, we find a monochromatic

$$X_{N,p} = \{b_1 + \sum_{i=1}^{N} \lambda_i a_i \mid |\lambda_i| \le p\} = b_1 + Z_{N,p}$$

where

$$Z_{N,p} = \{ \sum_{i=1}^{N} \lambda_i a_i \mid |\lambda_i| \le p\}.$$

Then in $Z_{N,p}$ we find a monochromatic $b_2+Z_{N',p}$ etc. Continuing in this manner we can find $b_1, b_2, \ldots, b_\ell$ such that the color of the sum

$$b_i + \sum_{j>i} \lambda_j b_j$$

depends only on i. For large enough ℓ, we may select m of these b_j to generate a monochromatic (m,p,1)-set.

This completes the case c = 1. For c > 1, a similar argument can be applied where, however, at each step p must be adjusted to compensate for the effect of c. □

CATEGORIES

The notion of a category having the Ramsey property was introduced by K. LEEB. It has been used to prove the Ramsey property for the category of finite vector spaces, among others. A category C is said to be *Ramsey* if for any objects A,B and number r, there is an object C such that for any r-coloring of the A-subobjects of C, all the A-subobjects of some B-subobject of C have the same color. Formally this says:

$$\forall A,B,r\ \exists C \ni \forall \binom{C}{A} \xrightarrow{f} [1,r],$$

$\exists$ a monomorphism, $B \xrightarrow{\phi} C$ and i such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}\binom{C}{A} & \xrightarrow{f} & [1,r] \\ \bar{\phi}\uparrow & & \uparrow \text{incl.} \\ \mathcal{C}\binom{B}{A} & \rightarrow & \{i\} \end{array} .$$

Here $\mathcal{C}\binom{C}{A}$ denotes the set of subobjects of C of isomorphism type A, and $\bar{\phi}$ the function induced by ϕ.

We could also abbreviate this by using the arrow notation. A category $\mathcal{C}$ is Ramsey if for every r and objects A,B there is an object C such that

$$C \rightarrow (\underbrace{B,B,\ldots,B}_{r}) .$$

To prove this property for a certain class of categories, including the category of sets (Ramsey's Theorem) and that of finite vector spaces, an elaborate induction is used. The induction is fundamentally determined by a generalization of the classical Pascal identity, $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$.

In his lecture notes on "Pascaltheorie", LEEB has developed more formally and generalized this kind of relationship and used it to prove some new Ramsey theorems, among other things. What we describe here is LEEB's generalization of the ordinary notion of labeled trees to that of trees labeled with objects from a category. A Ramsey theorem for these structures is then true if it was true in the original category.

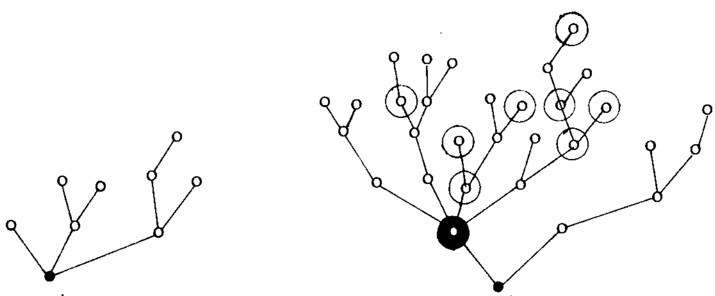
Consider a category $\mathcal{C}$. Then the category $\mathcal{O}rd(\mathcal{C})$ is defined to be the category of finite sequences of objects from $\mathcal{C}$. That is, the objects of $\mathcal{O}rd(\mathcal{C})$ are finite sequences of objects of $\mathcal{C}$, and morphisms $(C_1,C_2,\ldots,C_k) \rightarrow (D_1,D_2,\ldots,D_\ell)$, $k \le \ell$, are sequences $(\phi_1,\phi_2,\ldots,\phi_k)$ of morphisms from $\mathcal{C}$ such that $\phi_i\colon C_i \rightarrow D_{j(i)}$ for some $j(i)$, and $1 \le j(1) < j(2) < \ldots < j(k) \le \ell$.

We can define the category $\mathcal{T}rees(\mathcal{C})$ similarly. We consider rooted, labeled trees with an orientation, or ordering, of the branches at each vertex. We take the labels from the objects of $\mathcal{C}$. Morphisms are defined as follows. Let $\mathcal{T}_1,\mathcal{T}_2$ be two such objects, and let T_1,T_2 be their underlying rooted trees. First we "immerse" T_1 into T_2. An immersion $\psi\colon T_1 \rightarrow T_2$ is a monomorphic mapping from the vertices of T_1 to those of T_2 such that:

(a) For any two vertices x,y in T_1, $\psi(x \wedge y) = \psi(x) \wedge \psi(y)$, where for two vertices u,v in a rooted tree T, $u \wedge v$ denotes the last common vertex in the paths from the root to u and from the root to v, respectively.

(b) The order of the branches is preserved by ψ. That is, let $B_1,B_2,\ldots,B_k$ be the vertex sets of the branches at a vertex x in T_1, given in order, and let $D_1, D_2, \ldots, D_\ell$ be the vertex sets of the branches at $\psi(x)$ in T_2, given in order. Then for each i, $1 \le i \le k$, $\psi(B_i) \subseteq D_{j(i)}$ for some j(i), and $1 \le j(1) \le \ldots \le j(k) \le \ell$.

For example, the circled vertices in T_2 below indicate an immersion of T_1 into T_2:



Once we have an immersion ψ of T_1 into T_2, we then find a set of morphisms from $\mathcal{C}$ taking the labels from T_1 into the corresponding labels (by the immersion) of T_2. Such sets of morphisms of $\mathcal{C}$ (with restrictions determined by (a) and (b)) are defined to be the morphisms of $\mathcal{T}rees(\mathcal{C})$. If we denote a $\mathcal{C}$-labeled tree by [a,B], where a is the root label and B the sequence of branches at the root (with labels), we get the Pascal identity:

$$\mathcal{T}rees(\mathcal{C})\binom{[a,B]}{[c,D]} = \coprod_{B_i \in B} \mathcal{T}rees(\mathcal{C})\binom{B_i}{[c,D]} + \mathcal{C}\binom{a}{c} \times \mathcal{O}rd(\mathcal{T}rees(\mathcal{C}))\binom{B}{D}.$$

What this says is that every subtree (labeled) of type [c,D] in a tree of type [a,B] either has its root at the root of [a,B], or lies entirely in one of the branches at the root, with labels mapped accordingly. If one considers the identity for trees with only one branch at each point, and $\mathcal{C}$ the category with only a single object, then this identity becomes the classical Pascal identity.

We say that a category $\mathcal{C}$ is *directed* if for any objects A and B, for some object C there exist monomorphisms A → C and B → C. What LEEB proves is the following:

THEOREM. (LEEB). *If C is Ramsey and directed, then* $\mathcal{Trees}(C)$ *is Ramsey and directed.*

The proof uses the Ramsey property for C, together with the standard "product" argument, also used to prove (among other things) the result of GALLAI mentioned in the previous section.

A related and less complicated result, using the same basic techniques, is that if C is Ramsey and directed, then so is $\mathcal{Ord}(C)$.

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