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ON ADDRESSING GRAPHS WHICH HAVE SIMPLE SKELETONS

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INTRODUCTION

Let G be a finite, connected graph* with vertex set $V = V(G)$. For vertices v, v' in V, the distance in G between v and v', i.e. the minimum number of edges in any path between v and v', is denoted by $d_G(v,v')$. For the set $S = \{0,1,*\}$, define the distance $d^*(s,s')$ between two r-tuples $s = (s_1,s_2,\ldots,s_r) \in S^r$ and $s' = (s_1',s_2',\ldots,s_r')$ by

$$d^*(s,s') = \sum_{k=1}^{r} d(s_k,s_k')$$

where

$$d(s_k,s_k') = \begin{cases} 1 \text{ if } s_i = 0,\ s_k' = 1 \text{ or } s_k = 1,\ s_k' = 0, \\ 0 \text{ otherwise.} \end{cases}$$

A mapping $A : V \to S^r$ is said to be a valid addressing of G if

(1) $\quad d_G(v,v') = d^*(A(v),A(v'))$, all $v,v' \in V$.

Finally, the least value of r for which a valid addressing $A : V \to S^r$ exists for G is denoted by N(G).

The quantity N(G), which originally arose in connection with the investigation of certain data transmission schemes [7] has the subject of several recent studies [1], [2], [3], [4], [5]. A particularly appealing, though apparently difficult, conjecture is that for all G, the following upper bound on N(G) holds :

(2) $\quad N(G) \leq n-1$

where n denotes $|G|$, the number of vertices of G. It is known [1], [3], [4] that (2) holds when G is any complete graph, any complete bipartite graph, any tree, any cycle, any distance 2 graph and for several classes of graphs formed from smaller graphs satisfying (2).

In this note we show that (2) also holds for all graphs which have certain rather simple "strongly embedded" subgraphs, called skeletons . While the general validity of (2) for all connected graphs G remains open, the results given here may provide the first step in an inductive approach to this question.

A subgraph H of a graph G is said to be a skeleton of a provided :

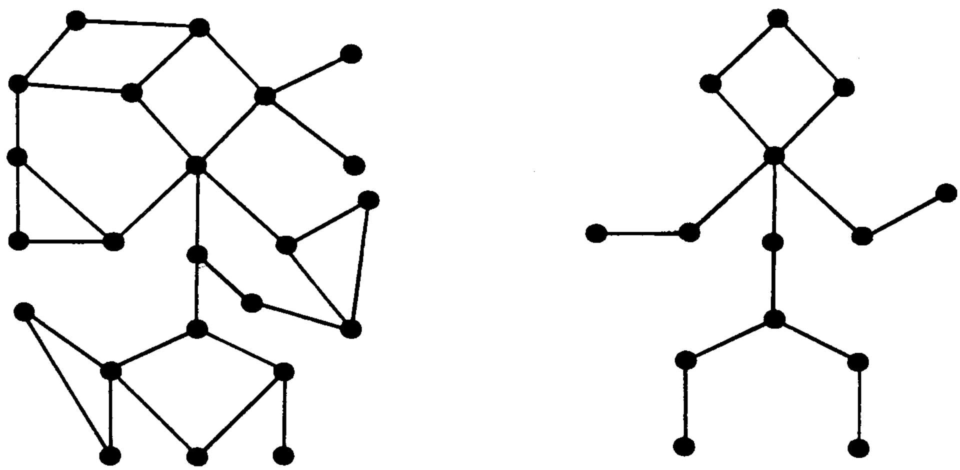
* For any undefined graph-theoretic terminology, see [6]

(i) For all $v \in V(G)$, either $v \in V(H)$ or there exists $v' \in V(H)$ with $d_G(v,v') = 1$.

(ii) For all v, $v' \in V(H)$, $d_H(v,v') = d_G(v,v')$.

Note that any skeleton of G must be an induced subgraph of G.

In Fig.1, we illustrate a graph G and a skeleton H of G.



G　　　　　　　　a sekleton of G

(a)　　　　　　　Fig. 1　　　　　　　(b)

The remainder of the paper is concerned with showing that G satisfies (2) whenever G has a skeleton H which is either a complete graph, a path or a star. In order to do this, we consider an equivalent problem involving the decomposition of a certain quadratic form associated with G.

The quadratic form Q(G)

Let $V = \{x_1, x_2, \ldots, x_n\}$ denote the vertex set of G. The quadratic form Q(G) is defined by

$$Q(G) = \sum_{1 \le i \le j \le n} d_{ij} x_i x_j$$

where $d_{ij} = d_G(x_i, x_j)$, $1 \le i, j \le n$. The following result appears in[4].

Fact. $N(G) \le r$ if and only if Q(G) can be written as

$$Q(G) = \sum_{k=1}^{r} \Big(\sum_{i \in A_k} x_i \Big) \Big(\sum_{j \in B_k} x_j \Big)$$

where $A_k, B_k \subseteq \{1,2,\ldots,n\}$.

In order to prove that (2) holds for the various graphs under consideration, we shall always make use of this equivalent formulation.

COMPLETE GRAPH SKELETONS

Theorem 1. If G has a skeleton which is a complete graph then G satisfies (2).

Proof : Let N denote the number of vertices of G and let H be a skeleton of G which is a complete graph on the vertices $x_1,\ldots,x_n$ of G where $n < N$. For H, the quadratic form Q(H) may be written

$$Q(H) = \sum_{1\le i\le j\le n} x_i x_j = \sum_{i=1}^{n-1} x_i(x_{i+1}+\ldots+x_n) \equiv \sum_{i=1}^{n-1} A_i B_i .$$

where $A_i = x_i$ and $B_i = \sum_{j=i+1}^{n} x_j$ for $1 \le i \le n-1$. We next modify A_i and B_i to form A_i' and B_i', $1 \le i \le n-1$, as follows :

For each vertex $x \in G\setminus H$, i.e., x is a vertex of G which is not a vertex of H :

(i) If $d_G(x,x_n) = 1$ then for each i, $1 \le i \le n-1$, with $d(x,x_i) = 2$, we add x to B_i.

(ii) If $d_G(x,x_n) = 2$ then let m denote the largest index such that $d(x,x_m) = 1$. Add x to A_m and, for each i, $1 \le i \le n-1$, with $d(x,x_i) = 2$, add x to B.

We claim that for a suitable choice of $C(x) = (x_{j_1}+\ldots+x_{j_s})$, and $x \in G\setminus H$, we can write Q(G) as

$$Q(G) = \sum_{i=1}^{n-1} A_i'B_i' + \sum_{x\in G\setminus H} x\cdot C(x) \tag{3}$$

Since the expansion (3) has only n-1 products then (2) will hold for G. To see that (3) is valid we must check several cases.

If $x \in G\setminus H$ and $x_i \in H$ then by the above rules xx_i appears in $\sum_{i=1}^{n} A_i'B_i'$ if and only if $d(x,x_i) = 2$. Thus, if we define each C(x) to contain the terms $\sum_{i=1}^{n} x_i$ then these distances will all be correct.

If x and x' are distinct vertices of $G\setminus H$, let S and S' be the sets of vertices of H at distance 1 from x and x', respectively. It follows from the definitions of A_i' and B_i' that the terms x and x' can appear on opposite sides in exactly one pair of terms $A_i'B_i'$ if $S \cap S' = \emptyset$, and, in any case, in no more than one pair.

The distance $d_G(x,x')$ may be 1,2 or 3. If $d_G(x,x') = 3$ then $S \cap S' = \emptyset$, the sum $\sum_{i=1}^{n-1} A_i'B_i'$ contains the term xx' exactly once and so, by defining C(x) to contain x' and C(x') to contain x, we have taken care of $d_G(x,x')$ in Q(G) in this case. If $d(x,x') = 1$ or 2, x and x' can appear on opposite sides in $A_i'B_i'$ at most once and the terms C(x) and C(x') can clearly be defined appropriately to match $d_G(x,x')$.

Thus we see that in all cases, the coefficient of xx' in Q(G) is exactly $d_G(x,x')$ for all vertices x, x' of G. This proves the theorem.

PATH SKELETONS

Theorem 2. If G has a skeleton which is a path then G satisfies (2).

Proof : As before, we assume G has N vertices and that H is a skeleton of G which is a path $(x_1,x_2,..,x_n)$, i.e., so that the only edges in H are $\{x_i,x_{i+1}\}$ for $1 \le i \le n-1$ where we assume $n < N$. If the edge $e = \{x_i,x_{i+1}\}$ is removed from H, the vertices of H are split into two disjoint sets A(e) and B(e) where $A(e) = \{x_j : 1 \le j \le i\}$, $B(e) = \{x_j : i < j \le n\}$. It is easily checked that

$$Q(H) = \sum_{e=\text{edge of } H} \left(\sum_{x \in A(e)} x \right) \left(\sum_{x' \in B(e)} x' \right) .$$

We make the following definitions for $y \in G \setminus H$:

$h(y) \equiv$ least integer such that $d_G(y, x_{h(y)}) = 1$,

$a(y) \equiv$ greatest integer $< h(y)$ such that

$$d_G(y, x_{a(y)}) \le d_G(x_{h(y)}, x_{a(y)}) .$$

$b(y) \equiv$ least integer $> h(y)$ such that

$$d_G(y, x_{b(y)}) < d_G(x_{h(y)}, x_{b(y)}) .$$

(It can happen for some y that a(y) or b(y) does not exist).

For $e = \{x_i,x_{i+1}\}$, define A'(e) and B'(e) as follows :

First, $A(e) \subseteq A'(e)$ and $B(e) \subseteq B'(e)$ for all e.

Also, if $y \in G \setminus H$, $x_{h(y)} \in A(e)$ and $i \neq b(y)-1$ then $y \in A'(e)$.

Similarly, if $y \in G \setminus H$, $x_{h(y)} \in B(e)$ and $i \neq a(y)$ then $y \in B'(e)$.

Write

$$\sum_{e=\text{edge of } H} A'(e)B'(e) = \sum_{x,x'} d'(x,x')xx' .$$

As in the proof of Theorem 1, it will suffice to show that

$$(4)\quad d_G(x,x') \geq d'(x,x') \geq \begin{cases} d_G(x,x') \text{ for } x,x' \in H, \\ d_G(x,x')-1 \text{ for } x \in H,\ x' \in H,\ x' \in G \setminus H, \\ d_G(x,x')-2 \text{ for } x,x' \in G \setminus H. \end{cases}$$

For if this is the case then by adjoining a sum of the form $\sum_{y \in G \setminus H} (y)C(y)$ for suitably chosen $C(y)$ we can bring all the deficient coefficients $d'(x,x')$ up to their required value of $d_G(x,x')$.

(i) If $x,x' \in H$ then

$$d_G(x,x') = d'(x,x')$$

is immediate from the definition of $A'(e)$ and $B'(e)$.

(ii) If $x_k \in H$, $y \in G \setminus H$ then certainly

$$d'(y,x_k) \leq d_G(x_{h(y)},x_k)$$

since y and $x_{h(y)}$ are never placed on opposite sides of a term $A'(e)B'(e)$. Also, it is immediate that

$$d'(y,x_k) \geq d_G(x_{h(y)},x_k) - 1$$

since we cannot have y missing from a term containing $x_{k(y)}$ twice while x_k is in the opposite term. Hence, (4) could only fail if either

$$d'(y,x_k) = d_G(x_{h(y)},x_k) = d_G(y,x_k) + 1$$

or

$$d'(y,x_k) = d_G(x_{h(y)},x_k) - 1 = d_G(y,x_k) - 2.$$

However, a careful consideration of the various possibilities (i.e., $a(y)$ and/or $b(y)$ does not exist, $k < a(y)$, $a(y) \leq k < k(y)$, etc.) shows that none of these violations can occur and so, (4) holds in this case.

(iii) If $y,y' \in G \setminus H$ then a similar computation verifies (4) in this case as well.

This completes the proof of Theorem 2.

CONCLUDING REMARKS

The following result can be proved in much the same way as Theorem 2.

Theorem 3. If G has a skeleton which is a star then G satisfies (2).

The reader may find it instructive to describe the appropriate modifications of the A(e) and B(e) for the decomposition of Q(H) which will work in this case.

It seems likely that a similar result holds as long as the skeleton of G is an arbitrary tree although at present we do not see how to do this. More generally, it may be true that one can show that if a graph G has a skeleton H satisfying (2) then G itself also satisfies (2).

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