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On the Distance Matrix of a Directed Graph

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ABSTRACT

In this note, we show how the determinant of the distance matrix $D(G)$ of a weighted, directed graph G can be explicitly expressed in terms of the corresponding determinants for the (strong) blocks G_i of G. In particular, when cof $D(G)$, the sum of the cofactors of $D(G)$, does not vanish, we have the very attractive formula

$$\frac{\det D(G)}{\operatorname{cof} D(G)} = \sum_i \frac{\det D(G_i)}{\operatorname{cof} D(G_i)}.$$

We consider finite directed graphs† G in which each (directed) edge e has associated with it an arbitrary non-negative "length" $w(e)$. For vertices v_i, v_j of G, the *distance* d_{ij} *from* v_i *to* v_j is defined by

$$d_{ij} = \min_{P(v_i, v_j)} w(P(v_i, v_j))$$

where $P(v_i, v_j)$ ranges over all directed paths from v_i to v_j and $w(P(v_i, v_j))$ denotes the sum of all edge-lengths in $P(v_i, v_j)$. We shall assume that G is *strongly connected* so that d_{ij} always exists. The *distance matrix* $D(G)$ of G is the square matrix which has d_{ij} as its (i, j) entry. This matrix, while not as common as the more familiar *adjacency matrix* of G, has nevertheless come up recently in several different areas, including communication

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† For graph theory terminology see Harary [8].

network design [5], graph embedding theory [3, 6, 7], molecular stability [9, 10], and network flow algorithms [1, 2].

In this note we study det $D(G)$, the *determinant* of the distance matrix of G. In particular, we derive an expression for det $D(G)$ which depends only on the (strongly connected) blocks of G and *not* on how they are interconnected. This gives perhaps the most natural explanation of the previously known result (see [5]) that for a uniform, undirected tree T_n on n vertices (i.e., all edges have length 1 in either direction),

$$\det D(T_n) = (-1)^{n-1}(n-1)2^{n-2}, \tag{1}$$

independent of the structure of T_n. We also establish a conjecture of one of the authors [9] for graphs having as blocks either cycles or single edges.

Before proceeding to the main result, we first require several preliminary ideas. For a square matrix A, let cof(A) denote the sum of the cofactors of A (cf. [4]). Form the matrix $\tilde{A}$ by subtracting the first row from all other rows, then the first column from all other columns and let $\tilde{A}_{11}$ denote the cofactor of $\tilde{A}$ in position (1, 1).

LEMMA

$$\operatorname{cof}(A) = \tilde{A}_{11}. \tag{2}$$

Proof. Let J be the matrix of 1's having the same order as A. If we write

$$\det(A + xJ) = c_0 + c_1 x, \tag{3}$$

it is obvious that

$$\operatorname{cof}(A) = c_1. \tag{4}$$

But if we let E_{11} denote the matrix with 1 in position (1, 1) and 0 everywhere else, then

$$\det(A + xJ) = \det(\widetilde{A + xJ}) = \det(\tilde{A} + xE_{11}). \tag{5}$$

Using (3) and (4), (5) implies (2). ∎

We may now state our main result. A *block* of a graph is defined to be a maximal subgraph having no cut points.

THEOREM. *If G is a strongly connected directed graph with blocks*

$G_1, G_2, \cdots, G_r$ *then*

$$\operatorname{cof} D(G) = \prod_{i=1}^{r} \operatorname{cof} D(G_i)$$

$$\det D(G) = \sum_{i=1}^{r} \det D(G_i) \prod_{j \neq i} \operatorname{cof} D(G_j). \tag{6}$$

Proof. We may select G_1 to be an *end block*, i.e., a block containing only one cut point of G (which we take to be labeled 0). Let $G_1^* = G-(G_1-\{0\})$ be the remainder of G. Note that the cut point 0 is *not* removed from G_1^*. We will first verify a decomposition in the form of (6) for G_1 and G_1^*. The theorem will then follow at once by induction by breaking down G_1^* successively until its blocks $G_2, \cdots, G_r$ are obtained. Assume $V(G_1)=\{0, 1, \cdots, m\}$ and $V(G_1^*)=\{0, m+1, \cdots, m+n\}$. Let

$$D(G_1) = \begin{pmatrix} 0 & a_1 & \cdots & a_m \\ b_1 & & & \\ \vdots & & E & \\ b_m & & & \end{pmatrix}, \qquad D(G_1^*) = \begin{pmatrix} 0 & f_1 & \cdots & f_n \\ g_1 & & & \\ \vdots & & H & \\ g_n & & & \end{pmatrix}.$$

Thus,

$$D(G) = \left(\begin{array}{c|c|c} 0 & \bar{a} & \bar{f} \\ \hline \bar{b} & E & b_i + f_j \\ \hline \bar{g} & g_i + a_j & H \end{array}\right).$$

Since $\det A = \det \tilde{A}$ then

$$\det D(G) = \det \left(\begin{array}{c|c|c} 0 & \bar{a} & \bar{f} \\ \hline \bar{b} & E-(b_i+a_j) & 0 \\ \hline \bar{g} & 0 & H-(g_i+f_j) \end{array}\right)$$

$$= \det \left(\begin{array}{c|c} 0 & \bar{a} \\ \hline \bar{b} & E-(b_i+a_j) \end{array}\right) \det (H-(g_i+f_j))$$

$$+ \det \left(\begin{array}{c|c} 0 & \bar{f} \\ \hline \bar{g} & H-(g_i+f_j) \end{array}\right) \det (E-(b_i+a_j))$$

$$= \det D(G_1) \operatorname{cof} D(G_1^*) + \det D(G_1^*) \operatorname{cof} D(G_1) \tag{7}$$

by the lemma. It also follows from the lemma that

$$
\begin{aligned}
\operatorname{cof} D(G) &= \det \left(\begin{array}{c|c} E-(b_i+a_j) & 0 \\ \hline 0 & H-(g_i+f_j) \end{array}\right) \\
&= \det (E-(b_i+a_j)) \det (H-(g_i+f_j)) \\
&= \tilde{D}(G_1)_{11} \tilde{D}(G_1^*)_{11} \\
&= \operatorname{cof} D(G_1) \operatorname{cof} D(G_1^*). \qquad (8)
\end{aligned}
$$

This completes the proof of (6) and the Theorem is proved. ∎

When none of the $\operatorname{cof} D(G_i)$ vanish, we can write $\det D(G)$ in the alternate form

$$\frac{\det D(G)}{\operatorname{cof} D(G)} = \sum_{i=1}^{r} \frac{\det D(G_i)}{\operatorname{cof} D(G_i)}. \qquad (9)$$

For the graph G_0 consisting of a single undirected edge of length 1 we have $D(G_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\operatorname{cof} D(G_0) = -2$, $\det D(G_0) = -1$. Thus, for a tree T_n with n vertices and $n-1$ undirected edges of unit length we have

$$\operatorname{cof} D(T_n) = (\operatorname{cof} D(G_0))^{n-1} = (-2)^{n-1},$$

$$\det D(T_n) = \operatorname{cof} D(T_n) \sum_{i=1}^{n-1} \frac{\det D(G_0)}{\operatorname{cof} D(G_0)} = (-2)^{n-1}(n-1) \cdot \frac{1}{2}$$

which implies (1).

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