$$h(x) = \det(x^2 I - 2Q^2) = \det \begin{bmatrix} x^2+2 & 2 \\ -2 & x^2 \end{bmatrix} = x^4 + 2x^2 + 4.$$

Thus $\sqrt{2}\omega$ is a root of $x^4 + 2x^2 + 4$. To find a polynomial $k(x)$ such that $k(\sqrt{2} - \omega) = 0$, we compute, using formula (2),

$$\begin{aligned} k(x) &= \det\left(x^2 \frac{p^{(2)}(Q)}{2} + xp^{(1)}(Q) + p(Q)\right) \\ &= \det(x^2 I + 2xQ + (Q^2 - 2I)) \\ &= \det \begin{bmatrix} x^2-3 & 2x-1 \\ -2x+1 & x^2-2x-2 \end{bmatrix} \\ &= x^4 - 2x^3 - x^2 + 2x + 7. \end{aligned}$$

Thus $\sqrt{2} - \omega$ is a root of $x^4 - 2x^3 - x^2 + 2x + 7$.

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Spectra of Numbers

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For a positive real number α, the sequence $S(\alpha)$, sometimes called the **spectrum** of α, is defined by

$$S(\alpha) = ([\alpha], [2\alpha], [3\alpha], \ldots)$$

where $[x]$ denotes the greatest integer $\leqq x$. The spectra of real numbers are known to have many interesting properties (cf. [7], [1], [2], [3], [9]). For example, for any three spectra $S(\alpha_1)$, $S(\alpha_2)$, $S(\alpha_3)$, some pair of them must have infinitely many elements in common.

In working with spectra, it is often useful to know whether a particular finite sequence is the beginning of some spectrum and, if so, what its possible continuations are. In this note we characterize the initial segments of spectra as sequences which are "nearly" linear. As a result, this characterization allows us to answer both of the preceding questions rather efficiently.

Let us call a finite sequence $A = (a_1, a_2, \ldots, a_n)$ **nearly linear** if all the following inequalities hold:

(1) $\max\{a_i + a_{k-i} : 1 \leqq i < k\} \leqq a_k \leqq 1 + \min\{a_i + a_{k-i} : 1 \leqq i < k\}$, for $1 < k \leqq n$.

For example, the sequences (2, 4, 6, 8, 11, 13, 15, 17, 20) and (1, 3, 4, 6, 7, 9, 11, 12, 14, 15, 17, 18) are nearly linear. Note that if α is an integer then $A = S(\alpha)$ is actually linear and $a_i + a_{k-i} = a_k$ for $1 \leqq i < k \leqq n$, and (1) holds trivially. Moreover, the initial segment $S_n(\alpha) = \{[\alpha], [2\alpha], \ldots, [n\alpha]\}$ of the spectrum of each real number α is also nearly linear: if $1 \leqq i < k \leqq n$, then $[k\alpha]$ equals either $[i\alpha] + [(k-i)\alpha]$ or $[i\alpha] + [(k-i)\alpha] + 1$, and so (1) holds. The main result of this note is the converse of this result.

THEOREM. *If $A = (a_1, a_2, \ldots, a_n)$ is nearly linear, then A is the initial segment $S_n(\alpha)$ for some α.*

Proof. The proof will be by induction on n. The result is clear for $n = 1$. Suppose the theorem holds for all values less than some integer $n > 1$ and let $A = (a_1, \ldots, a_n)$ be nearly linear. Since the sequence $A' = (a_1, \ldots, a_{n-1})$ is also nearly linear then by the induction hypothesis $A' = S_{n-1}(\alpha)$ for some α. This implies $a_k \leqq k\alpha < a_k + 1$ for $1 \leqq k < n$, which is equivalent to

$$L = \max_{1 \leqq k < n} \frac{a_k}{k} \leqq \alpha < \min_{1 \leqq k < n} \frac{a_k + 1}{k} = U.$$

The theorem will be proved if we can show

$$L^* = \max\left(L, \frac{a_n}{n}\right) < \min\left(U, \frac{a_n + 1}{n}\right) = U^*, \tag{2}$$

for then the desired α can be found just by choosing $L^* \leqq \alpha < U^*$.

There are two possibilities. First suppose $L^* = a_n/n$. By the definition of U, $U = (a_d + 1)/d$ for some d, $1 \leqq d < n$. By hypothesis, $a_{n-d}/(n-d) < (a_d + 1)/d$. Therefore, since A is nearly linear (use (1) with $k = n$),

$$L^* = \frac{a_n}{n} \leqq \frac{(a_d + 1) + a_{n-d}}{d + (n-d)} < \frac{a_d + 1}{d} = U$$

and consequently (2) holds.

Now, suppose $L^* = L$. Then for some d', $1 \leqq d' < n$, $L = a_{d'}/d'$. By hypothesis, $a_{d'}/d' < (a_{n-d'} + 1)/(n-d')$. Therefore, as above,

$$L = \frac{a_{d'}}{d'} < \frac{a_{d'} + (a_{n-d'} + 1)}{d' + (n-d')} \leqq \frac{a_n + 1}{n}$$

and, again, (2) holds. This proves the theorem.

In principle, a similar characterization should exist for nonhomogeneous spectra, i.e., sequences of the form $S(\alpha, \beta) = ([\alpha + \beta], [2\alpha + \beta], [3\alpha + \beta], \ldots)$. However, because it is possible for $[p\alpha + \beta] + [q\alpha + \beta]$ to equal $[(p+q)\alpha + \beta] - 1$, $[(p+q)\alpha + \beta]$ or $[(p+q)\alpha + \beta] + 1$, the conditions corresponding to (1) in this case will necessarily be more complicated (cf. **[8]**, **[6]**, **[10]**). In general, a much wider variety of behavior is possible using nonhomogeneous spectra. For example, in contrast to the intersection result (mentioned in the first paragraph) which holds for any three (homogeneous) spectra, it is possible to have infinitely many nonhomogeneous spectra which are pairwise mutually disjoint, e.g., $S(2^n, 2^{n-1})$, $n = 1, 2, 3, \ldots$.

We conclude with an interesting open question concerning nonhomogeneous spectra. An old result states that it is possible to partition the positive integers $\mathbb{Z}^+$ into two disjoint (homogeneous) spectra but that such a partition is impossible for three or more spectra (cf. **[4]**, **[5]**, **[7]**, **[9]**). However for any n, $\mathbb{Z}^+$ can be partitioned into n mutually disjoint nonhomogeneous spectra $S(\alpha_i, \beta_i)$, $1 \leqq i \leqq n$. Such decompositions are completely understood in the case that at least one α_k is irrational (cf. **[6]**); basically, all the $S(\alpha_i, \beta_i)$ are generated from some trivial decomposition of the form $\mathbb{Z}^+ = S(\alpha) \cup S(\beta)$ with $(1/\alpha) + (1/\beta) = 1$. On the other hand, if all the α_i are rational, then the situation is much less clear. The following striking conjecture has been proposed by Aviezri Fraenkel: If all the α_i are distinct and rational and $n \geqq 3$, then $\{\alpha_i : 1 \leqq i \leqq n\} = \{(2^n - 1)/2^i : 0 \leqq k < n\}$.

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The Psi Function

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The gamma function defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is well known as a generalized factorial because it satisfies the recurrence formula $\Gamma(x+1) = x\Gamma(x)$. It follows that $\Gamma(n) = (n-1)!$ for each positive integer n. The lesser known psi function, the subject of this paper, is defined as the logarithmic derivative of the gamma function:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

(The psi function and its derivatives ψ', ψ'', ψ''', etc., are also known as the digamma, trigamma, tetra-gamma, and pentagamma functions. Collectively they are called the polygamma functions.)

The integral which defines the gamma function converges for all $x > 0$. However the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ makes it possible to extend the definition of $\Gamma(x)$ to negative non-integral values of x. This extension means that the entire family of polygamma functions (that is, $\psi, \psi', \ldots, \psi^{(n)}, \ldots$) is defined for all real numbers except $0, -1, -2, \ldots$.

A recurrence formula for the psi function can be obtained by finding the logarithmic derivative of $\Gamma(x+1)$ in two different ways and then equating the results. The recurrence formula for $\Gamma(x)$ yields

$$\frac{d}{dx}\ln\Gamma(x+1) = \frac{d}{dx}[\ln x + \ln\Gamma(x)]$$
$$= \frac{1}{x} + \frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} + \psi(x),$$

whereas the definition of the psi function yields

$$\frac{d}{dx}\ln\Gamma(x+1) = \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \psi(x+1).$$