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COMBINATORIAL DESIGNS RELATED TO THE STRONG PERFECT GRAPH CONJECTURE*

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When α, ω are positive integers, we set $n = \alpha\omega + 1$ and look for zero-one matrices X, Y of size $n \times n$ such that

$$XY = YX = J - I, \qquad JX = XJ = \alpha J, \qquad JY = YJ = \omega J.$$

Simple solutions of these matrix equations are easy to find; we describe ways of constructing rather messy ones. Our investigations are motivated by an intimate relationship between the pairs X, Y and minimal imperfect graphs.

0. Introduction

Our graphs are "Michigan" except that they have vertices and edges rather than points and lines. If G is a graph, then $n = n(G)$ denotes the number of its vertices, $\alpha = \alpha(G)$ denotes the size of its largest stable (independent) set of vertices and $\omega = \omega(G)$ denotes the size of its largest clique. The graphs that we are interested in have the following three properties:

(i) $n = \alpha\omega + 1$,

(ii) every vertex is in *precisely* α stable sets of size α and in *precisely* ω cliques of size ω,

(iii) the n stable sets of size α may be enumerated as $S_1, S_2, \ldots, S_n$ and the n cliques of size ω may be enumerated as $C_1, C_2, \ldots, C_n$ in such a way that $S_i \cap C_i = \emptyset$ for all i but $S_i \cap C_j \neq \emptyset$ whenever $i \neq j$.

We shall call then (α, ω)-*graphs*. This concept, contrived as it may seem at first, arises quite naturally in the investigations of imperfect graphs; we are about to explain how.

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In the early nineteen-sixties, Claude Berge [1, 2] introduced the concept of a *perfect graph*: a graph is called perfect if and only if, for all of its induced subgraphs H, the chromatic number of H equals $\omega(H)$. Berge formulated two conjectures concerning these graphs:

(I) a graph is perfect if and only if its complement is perfect;

(II) a graph is perfect if and only if it contains no induced subgraph isomorphic either to a cycle whose length is odd and at least five or to the complement of such a cycle.

The concept of a perfect graph turned out to be one of the most stimulating and fruitful concepts in modern graph theory. The weaker conjecture (I), proved in 1971 by Lovász [10] became known as the Perfect Graph Theorem. The stronger conjecture (II), still unsettled, is known as the Strong Perfect Graph Conjecture.

A graph is called *minimal imperfect* if it is not perfect itself but all of its proper induced subgraphs are perfect. Clearly, every cycle whose length is odd and at least five is minimal imperfect, and so is its complement. The Strong Perfect Graph Conjecture asserts that there are no other minimal imperfect graphs. The first step towards a characterization of minimal imperfect graphs was made again by Lovász [11]: every minimal imperfect graph satisfies $n = \alpha\omega + 1$.

It follows from this that, in a minimal imperfect graph G,

for every vertex $v \in G$, the vertex set of $G - v$ can be partitioned into α cliques of size ω, and into ω stable sets of size α. (1)

Further refinements along this line are due to Padberg [12]: every minimal imperfect graph is an (α, ω)-graph. (Bland et al. [3] strengthened Padberg's result by proving that every graph satisfying (1) is an (α, ω)-graph.) Hence characterizing (α, ω)-graphs might help in characterizing minimal imperfect graphs.

It is easy to construct (α, ω)-graphs for every choice of α and ω such that $\alpha \geqslant 2$ and $\omega \geqslant 2$: begin with vertices $v_1, v_2, \ldots, v_{\alpha\omega+1}$ and join v_i and v_j by an edge if and only if $|i-j| \leqslant \omega - 1$, with subscript arithemetic modulo $\alpha\omega + 1$. The resulting graph, denoted by $C^{\omega-1}_{\alpha\omega+1}$, is an (α, ω)-graph. If $\omega = 2$, then $C^{\omega-1}_{\alpha\omega+1}$ is simply the odd cycle $C_{2\alpha+1}$; if $\alpha = 2$, then $C^{\omega-1}_{\alpha\omega+1}$ is the complement of the odd cycle $C_{2\omega+1}$. If $\alpha \geqslant 3$ and $\omega \geqslant 3$, then $C^{\omega-1}_{\alpha\omega+1}$ contains several pairs of nonadjacent vertices v, w such that joining v to w by an edge destroys no stable set of size α and creates no new clique of size ω. Hence the graph obtained by joining v to w is again an (α, ω)-graph. However, calling this graph "new" smacks of cheating: the structure of the largest stable sets and of the largest cliques has remained unchanged. To avoid such quibbling, we shall consider *normalized* (α, ω)-graphs in which every edge belongs to some clique of size ω. (As we shall see in a moment, every (α, ω)-graph contains a unique normalized (α, ω)-graph.) The purpose of this note is to present two different methods for constructing normalized (α, ω)-graphs other than $C^{\omega-1}_{\alpha\omega+1}$. The smallest of these graphs is the (3, 3)-graph shown in Fig. 1. (This graph and the (4, 3)-graph of Fig. 4 have been discovered independently by H.-C. Huang [3, 9].)

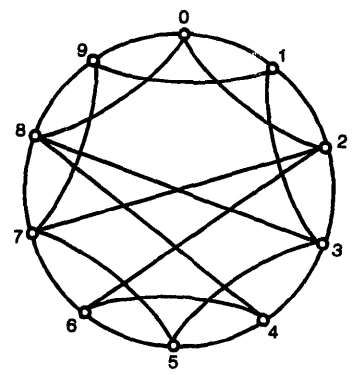

Fig. 1.

The problem of characterizing all the normalized (α, ω)-graphs can be given at least two additional interpretations. First, with each (α, ω)-graph we may associate two zero-one matrices X, Y of dimensions $n \times n$ such that the rows of X are the incidence vectors of the stable sets $S_1, S_2, \ldots, S_n$ and the columns of Y are the incidence vectors of the cliques $C_1, C_2, \ldots, C_n$. If I denotes the $n \times n$ identity matrix and if J denotes the $n \times n$ matrix filled with ones, then clearly

$$JX = XJ = \alpha J, \qquad JY = YJ = \omega J, \qquad XY = J - I. \tag{2}$$

In the terminology of Bridges and Ryser [4], the matrices X and Y form an "$(n, 0, 1)$-system on α, ω". Conversely, with each pair of zero-one matrices X, Y satisfying (2), we may associate a graph G with vertices $v_1, v_2, \ldots, v_n$ such that v_r is adjacent to v_s if and only if $Y_{rj} = Y_{sj} = 1$ for some j. Let us show that G is a normalized (α, ω)-graph. To begin with, each column of Y generates a clique of size ω in G and, since XY is a zero-one matrix, each row of X generates a stable set of size α in G. To show that G has no other cliques of size ω, consider an arbitrary clique of size ω and denote its incidence column vector by d. Clearly, Xd is a zero-one vector. In fact, since $J(Xd) = (JX)d = \alpha Jd$, the column vector Xd has $\alpha\omega = n - 1$ ones and one zero. Hence Xd is one of the columns of $J - I = XY$. Finally, since X is nonsingular, d must be a column of Y. A similar argument shows that every stable set of size α in G arises from some row of X. Hence G is an (α, ω)-graph; since each edge of G belongs to some clique of size ω, G is also normalized.

The matrix interpretation makes it easier to clarify the role of normalized (α, ω)-graphs. Consider an arbitrary (α, ω)-graph G and delete all those edges which belong to no clique of size ω. To show that the resulting graph H is an (α, ω)-graph, it will suffice to show that every stable set of size α in H was also stable in G. Beginning with G, define X and Y as above; in addition, let d denote the incidence row vector of an arbitrary stable set of size α in H. Since the cliques of size ω are the same in G and H, the row vector dY is zero-one. Since $(dY)J = d(YJ) = \omega dJ$, the vector dY is one of the rows of XY. Since Y is nonsingular, d is one of the rows of X, which is the desired conclusion. Hence H is the unique normalized (α, ω)-graph contained in G.

In the next section, we shall make use of the fact that the eqs. (2) imply

$$YX = X^{-1}XYX = X^{-1}(J-I)X = X^{-1}JX - I = J - I.$$

(The above observations are due to Padberg [12].)

Before proceeding, let us point out a simple fact which may be useful in constructing (α, ω)-graphs. For the moment, we shall refer to each pair of matrices (X, Y) satisfying (2) as a *solution*. Now, let r and s be positive integers such that $r+s=n$. Let A, A^* be $n\times r$ matrices, let B, B^* be $n\times s$ matrices, let C, C^* be $r\times n$ matrices and let D, D^* be $s\times n$ matrices. Finally, let us write

$$X_1=(A, B^*), \qquad X_2=(A^*, B), \qquad X_3=(A, B), \qquad X_4=(A^*, B^*)$$

and

$$Y_1=\begin{pmatrix} C \\ D^* \end{pmatrix}, \qquad Y_2=\begin{pmatrix} C^* \\ D \end{pmatrix}, \qquad Y_3=\begin{pmatrix} C \\ D \end{pmatrix}, \qquad Y_4=\begin{pmatrix} C^* \\ D^* \end{pmatrix}.$$

We claim the following:

if (X_1, Y_1), (X_2, Y_2), (X_3, Y_3) are solutions, then (X_4, Y_4) is a solution.

The proof is straightforward: since

$$X_1Y_1 = AC + B^*D^* = J - I,$$
$$X_2Y_2 = A^*C^* + BD = J - I,$$
$$X_3Y_3 = AC + BD = J - I.$$

we have $AC = A^*C^*$, $BD = B^*D^*$ and so

$$X_4Y_4 = A^*C^* + B^*D^* = J - I.$$

Similarly, the equations $JX_4 = X_4J = \alpha J$ and $JY_4 = Y_4J = \omega J$ follow quite routinely. It may be also interesting to note that:

if (Y_1, X_1), (Y_2, X_2), (Y_3, X_3) are solutions, then (Y_4, X_4) is a solution.

The point is that the equations

$$JY_k = Y_kJ = \alpha J, \qquad JX_k = X_kJ, \qquad Y_kX_k = J - I$$

imply $X_kY_k = J - I$ for each $k = 1, 2, 3$. Now $X_4Y_4 = J - I$ as above, and so $Y_4X_4 = J - I$.

An alternative interpretation of (α, ω)-graphs concerns a packing problem. With a slight abuse of the standard notation, let K_n denote the *directed* graph on n vertices such that, for every ordered pair of vertices v and w, there is a (unique) directed edge from v to w. Similarly, let $K_{\alpha,\omega}$ denote the complete bipartite graph in which each edge is directed from the α-set. As above, let n stand for $\alpha\omega + 1$. We claim that normalized (α, ω)-graphs correspond to partitions of the edge-set of K_n into n disjoint copies of $K_{\alpha,\omega}$. With every such partition, one may associate $n \times n$ matrices X, Y such that the jth column of X is the incidence vector of the α-set of the jth copy and such that the ith row of Y is the incidence vector of the ω-set of the ith copy. It is not difficult to verify that these matrices satisfy (2). Conversely, with every pair of zero-one matrices satisfying (2), one may associate a partition ot K_n into n disjoint copies of $K_{\alpha,\omega}$ by making the directed edge v_iv_j belong to the kth copy if and only if $x_{ik} = y_{kj} = 1$. Incidentally, if the directions of the edges are ignored, then these partitions become covers of the *undirected* K_n by n copies of *undirected* $K_{\alpha,\omega}$ such that each edge is covered precisely twice. Designs of this kind have been studied by C. Huang and Rosa [6, 7, 8].

Finally, let us return to the link between the problem of characterizing (α, ω)-graphs and the Strong Perfect Graph Conjecture: it is not clear that a solution to the former would indeed help to settle the latter. In fact, Tucker [13] succeeded in proving the Strong Perfect Graph Conjecture for all graphs G with $\omega(G) = 3$ without characterizing $(\alpha, 3)$-graphs. By virture of Padberg's theorem the Strong Perfect Graph Conjecture may be stated as follows:

> every (α, ω)-graph G with $\alpha \geqslant 3$ and $\omega \geqslant 3$ contains a smaller induced imperfect graph.

We shall say that an (α, ω)-graph G is of *type I* if it contains a set W of $\alpha + \omega - 1$ vertices such that $W \cap S \neq \emptyset$ for all stable sets of size α and $W \cap C \neq \emptyset$ for all cliques of size ω. Otherwise we shall say that G is of *type* II. It is easy to see that every (α, ω)-graph of type I contains a smaller induced imperfect graph (namely, the graph $G - W$ with $(\alpha - 1)(\omega - 1) + 1$ vertices and $\alpha(G - W) \leqslant \alpha - 1$, $\omega(G - W) \leqslant \omega - 1$). Hence the Strong Perfect Graph Conjecture would follow if every (α, ω)-graph with $\alpha \geqslant 3$ and $\omega \geqslant 3$ were of type I. Unfortunately, this is not the case: the (4, 4)-graph constructed in Section 2 of this paper is of type II. (In [5], it has been shown that every $C^{\omega-1}_{\alpha\omega+1}$ with $\alpha \geqslant 3$ and $\omega \geqslant 3$ is of type I.)

1. The first method

Each graph $C^{\omega-1}_{(\alpha+1)\omega+1}$ can be seen as arising from $C^{\omega-1}_{\alpha\omega+1}$ by a simple construction which, vaguely speaking, leaves most of the graph unchanged and increases the total number of vertices by ω. We are about to show that the same construction applies in a more general setting: if some set of $2\omega-2$ vertices of an (α,ω)-graph G induces a subgraph resembling a piece of $C_n^{\omega-1}$, then a simple local change in G creates an $(\alpha+1,\omega)$-graph H. More specifically, the properties required of the $2\omega-2$ vertices $v_1, v_2, \ldots, v_{2\omega-2}$ in G are that

each of the sets $C_k=\{v_{k+1}, v_{k+2}, \ldots, v_{k+\omega}\}$ with $k=0,1,\ldots,\omega-2$ is a clique,

and that

for each $k=2,3,\ldots,\omega-1$, either C_{k-1} is one of the α cliques partitioning $G-v_{k-1}$ or else C_{k-2} is one of the α cliques partitioning $G-v_{\omega+k-1}$.

The graph H has ω new vertices $a_1, a_2, \ldots, a_\omega$ in addition to the old $\alpha\omega+1$ vertices of G. The adjacencies in H are best described in terms of its cliques of size ω. First of all, we delete edges which belong to the $\omega-1$ cliques C_k specified above and no other cliques. Each C_k is replaced by *two* cliques,

$$C_k'=\{v_{k+1}, v_{k+2}, \ldots, v_{\omega-1}, a_1, a_2, \ldots, a_{k+1}\},$$
$$C_k''=\{a_{k+2}, a_{k+3}, \ldots, a_\omega, v_\omega, \ldots, v_{\omega+k}\}.$$

Finally, we introduce the clique $C^*=\{a_1, a_2, \ldots, a_\omega\}$. In the case $\omega=3$, the passage from G to H is schematically illustrated in Fig. 2.

Before proving that H is indeed an $(\alpha+1,\omega)$-graph, let us consider a few examples. To begin with, take $G=C_7^2$ and consider four consecutive vertices in the natural cyclic order. If these four vertices are labeled as v_1, v_2, v_3, v_4, then $H=C_{10}^2$; however, if they are labeled as v_1, v_3, v_2, v_4, then H is the graph of Fig. 1. Next, let G be the graph of Fig. 1. The three choices

$$(v_1, v_2, v_3, v_4)=(0,1,2,3),$$
$$(v_1, v_2, v_3, v_4)=(2,0,1\ 9),$$
$$(v_1, v_2, v_3, v_4)=(3,1,2,0)$$

lead to the $(4,3)$-graphs shown in Figs. 3, 4 and 5. These three graphs together with C_{13}^2 and the graph shown in Fig. 6 are in fact the only normalized $(4,3)$-graphs.

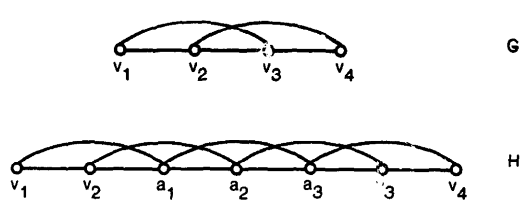

Fig. 2.

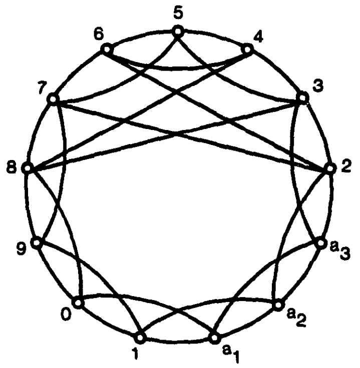

Fig. 3.

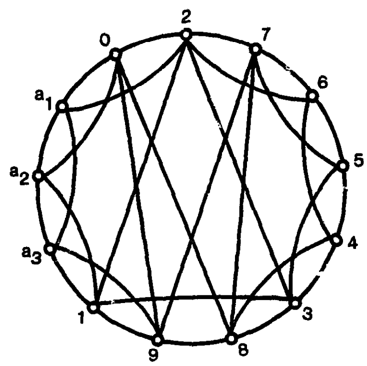

Fig. 4.

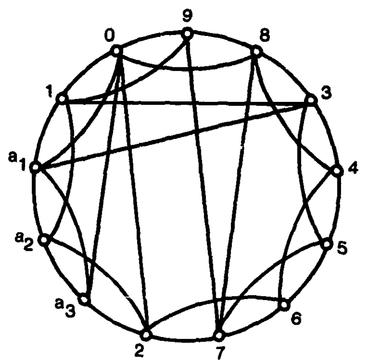

Fig. 5.

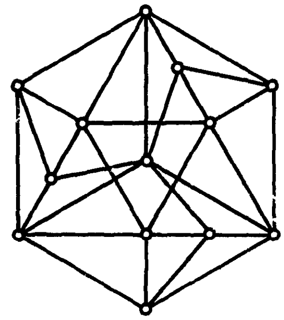

Fig. 6.

Now, let us establish that:

for every vertex $v \in H$, the vertex set of $H-v$ can be partitioned into $\alpha+1$ cliques of size ω. (3)

First, we consider the case $v \in G$. By (2), the vertex set of $G-v$ can be partitioned into α cliques of size ω. If one of these cliques is some C_k then replace this C_k by C_k' and C_k''; otherwise simply add C^* to the α cliques. Second, we consider the case $v \notin G$. Now $v = a_k$ for some k. If $1 < k < \omega$, then, by the assumption, either $C = C_{k-1}$ belongs to the partition of $G - v_{k-1}$ or else $C = C_{k-2}$ belongs to the partition of $G - v_{\omega+k-1}$. In either case, replacement of C by C_{k-2}' and C_{k-1}'' yields the desired partition of $G - a_k$. Finally, if $k=1$, then add C_k'' to the partition of $G - v_\omega$; if $k = \omega$, then add $C_{\omega-2}'$ to the partition of $C - v_{\omega-1}$.

With the help of (3), proving that H is an $(\alpha+1, \omega)$-graph becomes a routine matter. Let n stand for $(\alpha+1)\omega+1$ and let Y denote that $n \times n$ zero-one matrix whose columns are the incidence vectors of the $(\alpha-1)\omega+2$ cliques of size ω inherited by H from G and of the $2\omega-1$ new cliques C^*, C_k', C_k'', $0 \leq k \leq \omega-2$. By this definition and by construction of H, we have $JY = YJ = \omega J$. By (3), there is an $n \times n$ zero-one matrix X such that $YX = J - I$ and $JX = (\alpha + 1)J$. As we have seen in the preceding section, these equations imply $XY = J - I$. In addition,

$$XJ = \frac{1}{\omega} X(YJ) = \frac{1}{\omega}(J - I)J = \frac{n-1}{\omega} J = (\alpha + 1)J.$$

Since each edge of H belongs to some clique of size ω, the rows of X are the incidence vectors of stable sets. As in the preceding section, H had no other stable sets of size $\alpha + 1$. Hence H is an $(\alpha + 1, \omega)$-graph.

2. The second method

It seems that characterizing all the (α, ω)-graphs may be a rather difficult problem. At the moment, we cannot even characterize those (α, ω)-graphs which have circular symmetries. For these graphs, the associated matrices X, Y assume the form

$$X = \sum_{i \in A} Z^i, \qquad Y = \sum_{j \in B} Z^j$$

where Z is the permutation matrix of a cycle and

$$|A| \equiv \alpha, \qquad |B| \equiv \omega. \tag{4}$$

The condition $XY \equiv J - I$ reduces to

$$A + B \equiv \{1, 2, \ldots, \alpha\omega\} \tag{5}$$

with addition modulo $n \equiv \alpha\omega + 1$. The graphs $C^{\omega-1}_{\alpha\omega+1}$ correspond to, say, $A \equiv \{1, 2, \ldots, \alpha\}$ and $B \equiv \{0, \omega, 2\omega, \ldots, (\alpha - 1)\omega\}$. We are going to describe a more general class of solutions A, B to (4) and (5). Consequently, we shall obtain new (α, ω)-graphs with circular symmetries.

When $n - 1 = m_1 m_2 \cdots m_k$ for some integers m_i greater than 1, then we can consider the sets $M_1, M_2, \ldots, M_k$ defined by

$$M_i = \left\{ t \prod_{j=1}^{i-1} m_j : \quad 0 \leq t < m_i \right\}.$$

Clearly, $\sum_{i=1}^{k} M_i = \{0, 1, \ldots, n-1\}$. Now if $\prod_{i \in S} m_i = \alpha$ for some $S \subseteq \{1, 2, \ldots, k\}$, then

$$A = \sum_{i \in S} M_i, \qquad B = 1 + \sum_{i \notin S} M_i$$

satisfy (4) and (5).

For example, if $\alpha = \omega = 4$, then $n-1 = 2^4$ and so we consider

$$\{0, 1\}+\{0, 2\}+\{0, 4\}+\{0, 8\}=\{0, 1, \ldots, 15\}.$$

Now we might choose

$$A=\{0, 1\}+\{0, 2\}=\{0, 1, 2, 3\},$$
$$B=1+\{0, 4\}+\{0, 8\}=\{1, 5, 9, 13\}$$

but instead we shall choose

$$A=\{0, 1\}+\{0, 4\}=\{0, 1, 4, 5\},$$
$$B=1+\{0, 2\}+\{0, 8\}=\{1, 3, 9, 11\}.$$

The latter choice yields

$$X=Z^0+Z^1+Z^4+Z^5, \qquad Y=Z^1+Z^3+Z^9+Z^{11}.$$

The corresponding (4, 4)-graph G has vertices $v_0, v_1, \ldots, v_{16}$ such that v_i and v_j are adjacent if and only if

$$j-i \in \{2, 6, 7, 8, 9, 10, 11, 15\}$$

with arithmetic modulo 17. Clearly, this graph cannot be obtained by the method of the preceding section.

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